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GEOMETRIC STUDIES ON THE GLOBAL ASYMPTOTIC BEHAVIOUR OF DISSIPATIVE DYNAMICAL SYSTEMS

BY

MICHAEL YI LI ©

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THE UNDERSIGNED CERTIFIY THAT THEY HAVE READ, AND RECOMMEND TO THE FACULTY OF GRADUATE STUDIES AND RESEARCH FOR ACCEPTANCE, A THESIS ENTITLED **GOMETRIC STUDIES ON THE GLOBAL ASYMPTOTIC BEHAVIOUR OF DISSIPATIVE DYNAMICAL SYSTEMS** SUBMITTED BY **MICHAEL YI LI** IN PARTIAL FUFILMENT OF THE REQUIREMENTS FOR THE DEGREE OF **DOCTOR OF PHILOSOPHY** IN MATHEMATICS.

To the memory of my father: who was and will always be my inspiration for life;

to my wife: who is the fountain of all my creativity and imagination;

to Wang Linggai: who gives my first glimpse into the world of intellectuality.

ABSTRACT

In this thesis, some important questions concerning the global behaviour of the nonlinear dissipative dynamical systems in $\mathbf{R}^n$ are considered. These include the correlation between geometric properties (e.g. the Hausdorff dimension) of the global attractor, whose existence characterizes the dissipativity, and the dynamical behaviour of the system; lower and upper estimation for the Hausdorff dimension of the global attrractor; higher dimensional criteria of Bendixson and Dulac for the nonexistence of periodic solutions, and the problem of global stability. Dynamical systems having first integrals are also considered, in which case results are implied by much less restrictive conditions than hitherto possible, the relaxation being directly related to the number of independent first integrals. As an application of the general theory, the global stability of the endemic equilibrium of the SEIRS models with rather general nonlinear incidence rate in Epidemiology, which has been conjectured and remained open since 1986, is completely solved.

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Table of Contents

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

List of Figures

INTRODUCTION

In this introduction we give a brief outline of the thesis. We would like to do so by minimizing the amount of technicalities involved so that we may emphasize the various aspects which motivate the thesis. More detailed and specific introductions will be given at the beginning of each chapter.

The long-time behaviour of dissipative dynamical systems is characterized by the presence of a global attractor A towards which all trajectories converge. This is the largest compact invariant set in the phase space and thus contains the most important information regarding the asymptotic behaviour of the system. The structure of A may be very complicated even for a 3- dimensional system of ordinary differential equations with quadratic nonlinearities as in the case of the Lorenz system (see [8]). Considerable evidence from numerical analysis suggests that the Lorenz attractor may be a fractal set — a set whose Hausdorff and topological dimensions are different(see [20]). The complex structure of the global attractor is at least partially responsible for the 'chaotic' behaviour of the Lorenz system and some other dissipative systems. It is therefore of great interest to study the implications of the geometry of A on the dynamical behaviour of such systems.

Two monographs, Hale [5] and Temam [21], on dissipative dynamical systems have appeared recently. Both of them primarily deal with systems in infinite dimensional systems such as systems generated from differential equations with delays and from partial differential equations, where the structure of the global attractor becomes more complicated due to the infinite dimensionality of the phase space. The existence and different aspects of the global attractor are exploited, its finite Hausdorff dimensionality is proved in spite of the fact that the phase space is of infinite dimension. At about the same time, considerable efforts have been made to restrict the infinite dimensional system onto a finite dimensional invariant manifold

(the inertial manifold) which contains the global attractor and attracts exponentially the orbits outside, and thus transform the problem into a finite dimensional one (see [2]). Therefore it is of great importance to investigate the global behaviour of finite dimensional dissipative systems through its global attractor. It is in this area, this thesis tries to make a substantial contribution.

There are two types of dynamical systems that are of interest to us: the discrete system $\{T^k\}_{k\in\mathbf{N}}$ generated from the iteration of a nonlinear mapping $T : U \to \mathbf{R}^n$ where U is a open set of $\mathbf{R}^n$, and the continuous systems $\{\phi_t\}_{t\in\mathbf{R}}$ (or the flow) generated from an autonomous system of ordinary differential equations

$$x' = f(x) \tag{0.1}$$

where the mapping $x \mapsto f(x) \in \mathbf{R}^n$ is C^1 in some open set D of $\mathbf{R}^n$. The emphasis is on the latter. It is known that these two types of dynamical systems can behave quite differently. On the other hand, the time one map $T = \phi_1$ of the flow ϕ_t generates a discrete dynamical system. Therefore any result on the discrete systems can be readily applied to flows and hence to ordinary differential equations.

We start the thesis by introducing, in Chapter I, the definitions of these two types of systems and the related terminologies. After reviewing the usual concepts for orbits, limit sets and invariant sets, we introduce dissipativity through the existence of a compact *absorbing* set, and then define the global attractor as its ω-limit set. Some basic properties of the global attractor such as connectivity, attractivity and stability are also discussed. The main purpose of this chapter is to lay the groundwork and provide an easy reference for the development in later chapters. Most of the material in this chapter can be found in either [5] or [21].

Then, in Chapter II, we present a result which establishes a correlation between the Hausdorff dimension $\dim_H A$ of the global attractor A and the asymptotic behaviour of the dissipative system. Under certain connectivity assumptions on the domain U of the mapping T , we prove that if $\dim_H A < m + 1$ for some integer $m \geq 0$, the system $\{T^k\}_{k\in\mathbf{N}}$ can not possess certain m-dimensional structures

(the normal m-boundaries) invariant under T or otherwise. For example when $m = 1$, if the domain of T is simply connected, and $\dim_H A < 2$, then A contains no rectifiable Jordan curves, invariant or otherwise. This is a generalization of a result of R. A. Smith [17], who showed that A can not contain an invariant Jordan curve.

This type of results can be useful in two different ways: on the one hand, it gives us a way of obtaining lower estimates for the Hausdorff dimension of the global attractor which is a well-known difficult problem. Suppose that D is convex or there is a convex bounded absorbing set. By detecting the existence in A of a normal m-boundary, invariant or not, our results will imply that $\dim_H A \geq m+1$. The only other approach to this problem we are aware of is given by R. Temam and his colleagues (see [21], Chapter VII). Their method is to estimate the dimension of unstable manifolds of hyperbolic equilibria; on the other hand, these results, when applied to differential equations, give a weak condition precluding the existence of some invariant structures such as periodic orbits and higher dimensional invariant tori. The case $m = 1$ will be used in Chapter IV to derive Bendixson's criterion in $\mathbf{R}^n$.

Our next subject deals with the problem of global stability for autonomous system of ordinary differential equations. This is covered in Chapters III, IV, and V. The stability problem is the most fundamental in the qualitative studies of differential equations. Let $x = x(t, x_0)$ denote a solution to (0.1) such that $x(0, x_0) = x_0$. Then $x(t, x_0)$ is said to be *stable*, if for all $\bar{x}$ sufficiently close to x_0, the solution $x(t, \bar{x})$ stays arbitrarily close to $x(t, x_0)$ for all $t > 0$. Sometimes a stronger version, asymptotical stability, is useful which also require that all such nearby solutions $x(t, \bar{x})$ satisfy $x(t, \bar{x}) - x(t, x_0) \to 0$ as $t \to \infty$. It is of great interest both in theory and in practice that we find out how far away $\bar{x}$ can be from x_0 so that these convergence properties of $x(t, \bar{x})$ still hold. If they hold in the region being considered, we say $x(t, x_0)$ is globally asymptotically stable, or simply globally stable. Most of the interest so far is on the global stability of an equilibrium solution, namely a solution satisfying $x(t, x_0) = x_0$ for all t, which is the most

simple case. Even in this simple case the question is still very difficult to solve, in part because of the lack of tools. The most common method is construction of the Lyapunov functions. A Lyapunov function is a real valued function $x \mapsto V(x)$ which decreases along each solution to (0.1). The application of this very useful idea is often hindered by the fact that, in many cases, Lyapunov functions are very difficult to construct and there is practically no general way such a function may be constructed.

Poincaré and Bendixson found a different route to this problem when the dimension of the system is 2. In what is now known as the Poincaré - Bendixson theory (see [6]), they proved that, when $n = 2$, the omega limit set of any bounded trajectory to (0.1) either contains an equilibrium or is a periodic orbit. Therefore, if $\bar{x} \in D$ is the unique equilibrium of (0.1), which is also locally stable and (0.1) is dissipative, to show $\bar{x}$ is globally stable in D is equivalent to show (0.1) has no nonconstant periodic solutions. Bendixson [1] also shows that if $\mathrm{div} f < 0$ everywhere in $\mathbf{R}^2$, then (0.1) has no nonconstant periodic solutions. This is now what we call the Bendixson's criterion. Dulac [4] generalized this to the following: if D is a simply connected and $\mathrm{div}(\alpha f) < 0$ in D for some real-valued function $\alpha > 0$, then (0.1) has no closed path in D. Note that the Dulac's conditions provide more flexibility by introducing the arbitrary function α. Therefore, for planar systems, the problem of global stability can be solved by verifying that conditions of Bendixson or Dulac hold in D.

Unfortunately, the theory of Poincaré and Bendixson does not hold for general autonomous systems of dimension higher than 2, where more complicated dynamics are possible and even chaos may be present, as in the case of the Lorenz system. A lot of effort has been spent on special types of systems which behave like or can be reduced to planar systems. For example, M. Hirsch and H. L. Smith proved that 3-dimensional monotone systems satisfy the Poincaré-Bendixson property (see the survey paper of H. L. Smith [19] for references). R. A. Smith [18] has given some conditions under which a n-dimensional system can be projected onto a planar system. However, results for general systems are very scarce.

If we look at the problem a little more closely, we can see that the first thing we need to do in order to show global stability of the unique equilibrium is to prove the nonexistence of nonconstant periodic solutions, since they are bounded solutions which do not approach the equilibrium. This requires generalizations of the criteria of Bendixson and Dulac to systems of arbitrary dimension. The research on this line started only very recently. Works with full generality are due to R. A. Smith [17] and J. S. Muldowney [10]. Smith proves that if $\lambda_1 + \lambda_2 < 0$ everywhere in a simply connected open set D, where $\lambda_1 \geq \cdots \geq \lambda_n$ are eigenvalues of $\frac{1}{2}(\frac{\partial f}{\partial x}^* + \frac{\partial f}{\partial x})$, then no piece-wise smooth Jordan curves can be invariant with respect to (0.1). Muldowney proves a similar result under a more general condition $\mu(\frac{\partial f}{\partial x}^{[2]}) < 0$ in D, where $\frac{\partial f}{\partial x}^{[2]}$ is the second additive compound matrix of the Jacobian matrix $\frac{\partial f}{\partial x}$, and μ is the Lozinskiĭ measure, or the logarithmic norm, corresponding to a general vector norm in $\mathbf{R}^N$, $N = \binom{n}{2}$. If the norm is euclidean, Muldowney's condition gives rise to that of Smith's. When $n = 2$, both their results reduce to Bendixson's criterion.

The second obstacle to proving the global stability is that, for higher dimensional systems, the omega limit set of a bounded trajectory which contains no equilibria may be more complicated than a Jordan curve. Thus a bounded trajectory may not converge to the unique equilibrium $\bar{x}$ even in the absence of periodic orbits. The insight of R. A. Smith shows us his higher dimensional generalization of Bendixson's criterion implies all bounded trajectories converge to certain equilibria. In particular, if there is a unique equilibrium $\bar{x}$, Smith's condition $\lambda_1 + \lambda_2 < 0$ holds everywhere in the simply connected region D would imply that $\bar{x}$ is globally stable in D. The proof of Smith's result uses the C^1 Closing Lemma of Pugh.

Smith's result will be expanded and generalized in this second part of the thesis. Our first goal is to obtain, in Chapter IV, higher dimensional generalizations of Dulac's criteria. Since this is a problem of great importance for its own right, we discuss it in the context of both dissipative and nondissipative systems. For dissipative systems, this is done by deriving concrete conditions which imply the Hausdorff dimension of the global attractor $\dim_H A < 2$, by our main result in

Chapter II. We devote the whole Chapter III to the upper estimation of $\dim_H A$. For nondissipative systems, this is achieved by considering general functionals defined on the rectifiable 2-surfaces with the same boundary, and their evolution properties under the dynamics of (0.1). This is a generalization of Muldowney's method, which considers the areas of such surfaces. We prove that if D is simply connected, and if for some $N \times N$ matrix-valued function $x \mapsto A(x)$ which is nonsingular and C^1 in D, and for some Lozinskiĭ measure μ in $\mathbf{R}^N$, $N = \binom{n}{2}$,

$$\mu\left(A_f A^{-1} + A \frac{\partial f}{\partial x}^{[2]} A^{-1} \right) < 0 \tag{0.2}$$

holds everywhere in D, where A_f denotes the directional derivative of A in the direction of f, then no simple closed rectifiable curves can be invariant with respect to (0.1). In particular, under this condition, (0.1) can not have orbits of the following types: (a) periodic orbits; (b) homoclinic orbits; (c) a pair of heteroclinic orbits of the same equilibria; (d) heteroclinic cycles, since each case will give rise to a simple closed invariant curve which is also rectifiable. When $A = I$, (0.2) reduces to the condition of Muldowney. We would also like to note that the theory of compounded equations, which has matured only recently through the efforts of Schwarz [16], Muldowney [10], [11], [12], and Li and Muldowney [7], has been instrumental in this study. Then In chapter V, using Pugh's C^1 Closing Lemma [13], [14], [15] and the Centre Manifold Theorem [6], we prove that the Dulac type condition (0.2) and its weaker forms imply that bounded trajectories converge to equilibria. Moreover, we prove that these conditions also have severe restrictions on the structure of compact invariant sets. These Autonomous Convergence Theorems, following R. A. Smith, enable us to develop a new geometric approach to the problem of global stability for higher dimensional autonomous systems of ordinary differential equations.

In the third part of the thesis, which is contained in Chapter VI, we discuss autonomous systems having first integrals. Existence of first integrals often indicates the presence of certain physical conservation laws manifested in the system. Mathematically speaking, a first integral of (0.1) is a nonconstant real-valued function $H(x)$ which is constant along each solution. Thus every solution to (0.1)

stays on a lower dimensional invariant manifold defined by an equation $H(x) = c$ with c being determined by its initial value. This means that, in the presence of first integrals, systems in $\mathbf{R}^n$ are only capable of displaying behaviour which is typical of systems in lower dimensions. We prove that, in this case, the results obtained in earlier chapters can now be proved under considerably less restrictive conditions. For example, the Bendixson Criterion $\mu\big(\frac{\partial f}{\partial x}^{[2]}\big)$ for general systems in Chapter IV can be replaced by $\mu\big(\frac{\partial f}{\partial x}^{[r+2]}\big)$ if the system has r independent first integrals. This result will play an important role in Chapter VII where we resolve some hitherto unsolved problems in Mathematical Biology.

A traditional approach to systems with first integrals is to employ the equation $H(x) = c$ to reduce the number of variables, and thus reduce the dimension of the problem. This often relies critically on the choice of coordinates in the invariant manifold. In our study, the focus is on the implications for the linear variational equations. This leads to some new discoveries of the underlining geometry and novel techniques.

A conscious effort has been made throughout the thesis to illustrate the general theory by meaningful examples. Different aspects of the Lorenz system are investigated in the corresponding chapters, making new contributions to the understanding of its complicated dynamics. A variety of concrete models from different areas are given when each new class of systems is introduced in Chapter VI, providing motivations and insights. Moreover, in Chapter VII, our new geometric approach to global stability developed in Chapters III, IV, and V, as well as the study on the systems with first integrals in Chapter VI is applied to Mathematical Biology, where we are able to resolve the question of global stability of some epidemiological models, which has been a long-standing open problem in this area. We have reason to believe that this new approach will also find important applications in other areas.

To conclude this introduction, we would like to make some notes on the notations used in the thesis: capital letters A, B, D, X, Y, etc. are usually used to denote matrices or sets. When there is a conflict between matrices and sets,

letters for sets are bold-faced. For example, the global attractor is denoted by A in Chapters I and II, but is denoted by a bold face $\mathbf{A}$ in Chapters III and IV, and the letter A is reserved for matrices; lower case letters a, b, x, y, etc usually denote variables in the phase space; calligraphic letters $\mathcal{A}$, $\mathcal{S}$, etc. are exclusively reserved for functionals.

The sections are numbered within each chapter and subsections within each section. For example, §6.2 represents Section 2 in Chapter VI, and §4.2.1 indicates the first subsection of Section 2 in Chapter IV. Theorems, propositions, lemmas, and corollaries are numbered together within each section. For example, Theorem 3.4.2 is in the section §3.4 of Chapter III.

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CHAPTER I

DISSIPATIVE DYNAMICAL SYSTEMS

Since dissipative dynamical systems are the main subject of investigation in this thesis, it is necessary to lay the groundwork before we proceed to detailed discussions. This will be the purpose of the present chapter. We will carefully define and discuss some key concepts and terminologies that are closely related to later discussions and will be referred to throughout.

The most important concept — the concept of dissipativeness — is introduced through absorbing sets, and dissipative dynamical systems are characterized by the existence of a global attractor — the maximal compact invariant set which attracts all bounded sets. Much attention will be devoted to some basic properties of the global attractor for the simple reason that most interesting phenomena appear on or near the global attractor for dissipative systems (See Hale [1] and Temam [4]).

Two types of dynamical systems are of interest: (a) discrete dynamical systems generated by iteration of mappings from $\mathbf{R}^n$ to $\mathbf{R}^n$ (discussed in the section §1.1); (b) continuous dynamical systems generated from autonomous ordinary differential equations in $\mathbf{R}^n$ (discussed in the section §1.2). Even though the two are closely interrelated, they can behave very differently. This is the main reason that we opt to treat them separately. A drawback of such a treatment is that discussions in the section §1.1 and §1.2 appear parallel. We adopt this approach for easy reference later.

A periodic system of ordinary differential equation in $\mathbf{R}^n$ can generate either a discrete dynamical system by the Poincaré map or a continuous dynamical system on a product space, which is also called the skew-product flow (See Hale [1]). Since both approaches will be employed in later chapters, they are discussed in full detail in the section §1.3.

In section §1.4, as an illustration, the Lorenz model is analyzed. We will show,

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by finding a bounded absorbing set, that the Lorenz equation is dissipative and thus possesses a global attractor. This model will be used as an example in later chapters where different aspects of this Lorenz attractor are discussed.

§1.1. Discrete Dynamical Systems

In this section we discuss the discrete dynamical system $\{T^k\}_{k\in\mathbf{N}}$ generated from the iteration of a mapping $T : \mathbf{R}^n \to \mathbf{R}^n$. We define some key concepts and terminologies such as orbits, limit sets, invariant sets, attracting, absorbing, etc. We focus on the concepts of dissipativeness and global attractor as well as basic properties of the global attractor such as existence, connectivity, stability and attractivity.

Suppose $U \subset \mathbf{R}^n$ is an open set, and $T : U \to \mathbf{R}^n$ is a continuous mapping such that $T(U) \subset U$. Let $T^0 = \mathrm{id}$, the identity mapping in $\mathbf{R}^n$. Define recursively $T^k = T(T^{k-1})$, $k \geq 1$. We call $\{T^k\}_{k=1}^{\infty}$ the *discrete* dynamical system generated by the mapping T.

For each $x \in T$, the *positive semi-orbit* $\gamma^+(x)$ through x is defined as $\gamma^+(x) = \cup_{k\geq 0} T^k x$. A *negative semi-orbit* through x is a sequence $\{x_j : j = 0, -1, -2, \cdots\}$ such that $x_0 = x$, $Tx_{j-1} = x_j$ for all j. An *orbit* through x is a sequence $\{x_j : j = 0, \pm 1, \pm 2, ...\}$ such that $x_0 = x$, $Tx_{j-1} = x_j$ for all j.

Since T is not assumed to be one to one, there may be several orbits through the same point. Moreover, since $T(U)$ may not be equal to U, negative semi-orbits of some points may not even exist.

A subset $S \subset U$ is said to be *invariant* under T if, for each $x \in S$, a complete orbit through x exists and is contained in S. It is *positively invariant* under T if $\gamma^+(x)$ exists and is contained in S for all $x \in S$. It is easy to see that S is positively invariant if and only if $T\,S \subset S$; it is invariant if and only if $T\,S = S$. The following are examples of invariant sets:

(1) A fixed point $\{x_0\}$; namely $T\,x_0 = x_0$.

(2) The periodic orbit $\{T^k x_0 : 1 \leq k \leq m\}$ of a m-periodic point x_0; namely $T^m\,x_0 = x_0$.

(3) A full orbit of any point.

The most important invariant sets are limit sets. For any subset $B \subset U$, the ω-limit set $\omega(B)$ and the α-limit set $\alpha(B)$ of B are defined as

$$\omega(B) = \cap_{j\geq 0} \operatorname{cl} \cup_{k\geq j} T^k B, \tag{1.1}$$

and

$$\alpha(B) = \cap_{j\geq 0} \operatorname{cl} \cup_{k\geq j} H(k, B), \tag{1.2}$$

respectively, where

$$H(k, B) = \cup_{x\in B} H(k, x)$$

and

$$H(k, x) = \{ y \in U \ : \text{ there is a negative semi-orbit } \{x_{-i}\}_{i=1}^{\infty} \text{ through } x \text{ such that } x_{-k} = y \}.$$

We characterize the limit sets in the following proposition whose proof can be found in [1] or [4].

Proposition 1.1.1. *Suppose $x \in U$ and $B \subset U$. Then*

(1) $y \in \omega(x)$ *if and only if there exists a sequence of integers* $n_j \to \infty$ *such that* $T^{n_j} x \to y$ *as* $n_j \to \infty$.

(2) $y \in \alpha(x)$ *if and only if there exists a sequence of integers* $n_j \to \infty$ *and a sequence of points* y_{n_j} *such that* $T^{n_j} y_{n_j} = x$ *and* $y_{n_j} \to y$ *as* $j \to \infty$.

(3) $y \in \omega(B)$ *if and only if there exists a sequence of integers* $n_j \to \infty$ *and a sequence of points* $y_{n_j} \in B$ *such that* $T^{n_j} y_{n_j} \to y$ *as* $n_j \to \infty$.

(4) $y \in \alpha(B)$ *if and only if there exists a sequence of integers* $n_j \to \infty$ *and a sequence of points* $y_{n_j} \in U$ *such that* $T^{n_j} y_{n_j} \in B$ *for all* n_j *and* $y_{n_j} \to y$ *as* $n_j \to \infty$.

Let $|\cdot|$ denote the vector norm in $\mathbf{R}^n$. Define

$$\mathrm{d}(x, A) = \inf_{y\in A} |x - y| \tag{1.3}$$

and

$$d(B, A) = \sup_{x \in B} d(x, A), \tag{1.4}$$

for any two subsets A and B of $\mathbf{R}^n$. A subset A is said to *attract* subset B if

$$d\big(T^n(B), A\big) \to 0 \quad \text{as} \quad n \to \infty. \tag{1.5}$$

The following lemma establishes the invariance of limit sets. For its proof, we refer the reader to [1] or [4].

Lemma 1.1.2. *Suppose B is a nonempty subset of U such that $\cup_{x \in B} \gamma^+(x)$ is bounded. Then $\omega(B)$ is nonempty, compact, and invariant under T. Moreover, $\omega(B)$ attracts B. If $\cup_{k \geq 0} H(k, B)$ is nonempty and bounded then $\alpha(B)$ is nonempty, compact and invariant under T.*

Corollary 1.1.3. *For any $x \in U$, if $\gamma^+(x)$ is bounded then $\omega(x)$ is nonempty, compact, and invariant; if $\cup_{k \geq 0} H(k, x)$ is nonempty and bounded then $\alpha(x)$ is nonempty, compact, and invariant.*

A subset $B \subset \bar{B} \subset U$ is said to be *absorbing* if each compact subset B_0 of U satisfies $T^k(B_0) \subset B$ for all sufficiently large k. We say the discrete dynamical system $\{T^k\}_{k=1}^{\infty}$ is *dissipative*, or simply T is *dissipative*, if there is in U a bounded absorbing set.

A global attractor for the mapping T is a nonempty compact set $A \subset U$ such that A is invariant under T and attracts every compact subset of U. It follows that such a set is necessarily unique. Moreover, it is the maximal compact invariant set in U.

Remark. For dynamical systems in a space of infinite dimension, an absorbing set is usually required to 'absorb' every bounded subset; namely, the subset B_0 in the definition is required to be bounded (see [1] and [4]). Since compact set is necessarily bounded, this definition is stronger than the one we give here, when the phase space is infinite dimensional. Since $\mathbf{R}^n$ is locally compact, the two definitions are equivalent in the context of the present chapter when $U = \mathbf{R}^n$.

However, when U is a bounded subset of $\mathbf{R}^n$, our definition is more appropriete. The same remark also applies to the global attractor.

Suppose that the global attractor A for T exists. Then T is dissipative; namely, there is a bounded absorbing set $B \subset U$. In fact, if $V \subset U$ is a neighbourhood of A, then for each compact subset $B_0 \subset U$, $T^k(B) \subset V$ when k sufficiently large. Therefore V is absorbing.

Conversely, as the following result demonstrates, a dissipative system always possesses a global attractor.

Theorem 1.1.4. *Suppose that T is dissipative with a bounded absorbing set B. Then $A = \omega(B)$ is the global attractor for T.*

Proof. First of all, since B is bounded, $\bar{B}$ is compact. B attracts $\bar{B}$. Thus $\gamma^+(B)$ is bounded. It then follows from Lemma 1.1.2 that $\omega(B)$ is nonempty, compact and invariant. Moreover, $\omega(B)$ attracts B. The orbit of every compact subset of U eventually enters and stays in B and is therefore attracted by $\omega(B)$ as well. $\square$

Remarks.

(i). The above discussion demonstrates that the dissipativity of T can be characterized by the existence of the global attractor.

(ii). If B is the absorbing set in U, we actually have $A = \omega(B) = \cap_{k=1}^{\infty} T^k(B)$. In fact, since $T^k(B) \subset B$ for all sufficiently large k, we know $\omega(B) \subset B$. On the other hand, it follows from $T^k(B) \subset \operatorname{cl} \cup_{j \geq k} T^j(B)$ that $\cap_{k=1}^{\infty} T^k(B) \subset \omega(B)$.

(iii). The global attractor $A = \omega(B)$ does not depend on the particular adsorbing set B because of its attractivity.

Our next result concerns the connectivity of the global attractor.

Proposition 1.1.5. *If there is a bounded connected absorbing set, then the global attractor is also connected.*

Proof. Suppose B is a bounded absorbing set and is connected. Then $A = \omega(B) \subset B$ is the global attractor from Theorem 1.1.4. If A is not connected, then there are open sets V and W such that $V \cap W = \emptyset$, $A \subset V \cup W$ and $A \cap V \neq \emptyset$, $A \cap W \neq \emptyset$. By the continuity of T, $T^k(B)$ is connected for all $k \geq 0$. Moreover, $V \cap T^k(B) \neq \emptyset$ and $W \cap T^k(B) \neq \emptyset$ since $A \subset T^k(B)$ for all $k \geq 0$. Therefore there is a $x_k \in T^k(B) \setminus V \cup W$. The sequence $\{x_k\}_{k=1}^{\infty}$ is contained in B and thus is bounded and we may assume that $x_k \to x \in A = \omega(B)$. Clearly $x \notin U \cup V$, which is a contradiction. □

In what follows, we will show that the global attractor, whenever it exists, has the strongest stability. We say an invariant set $K \subset U$ is *stable* if, for any neighbourhood V of K, there is a neighbourhood W of K such that $T^k(W) \subset V$ for all $k \geq 0$. K is *asymptotically* stable if it is stable and attracts every point in a neighbourhood. We say K is *uniformly asymptotically stable* if it is stable and attracts a neighbourhood.

Remarks.

(i). In the above definition of stability for compact invariant set K , let $V' = \cup_{k \geq 0} T^k(W)$; then $V' \subset V$ and $TV' \subset V$. Therefore, the stability of K can be characterized as that each neighbourhood V of K contains a neigbouhood V' which is positively invariant.

(ii). As a result of (i), for a stable compact invariant set K, if K attracts a point $\bar{x}$, it attracts a neighbourhood of $\bar{x}$. Therefore, by continuity of T, K attracts every point of U implies K attracts every compact subset of U . Since $\mathbf{R}^n$ is locally compact, this implies K attracts a neighbourhood. We therefore arrive at an important conclusion, for a compact invariant set, asymptotic stability is equivalent to uniformly asymptotic stability. It is also very important to note that this equivalence no longer holds for dynamical system in an infinite dimensional space (see Hale [1]). This is one of the remarkable difference between dynamical system in finite and infinite dimensional spaces.

We have seen that the global attractor attracts every compact subset of U. Therefore it attracts a bounded neighbourhood of itself. As we shall see in the following result, it is actually uniformly asymptotically stable.

Theorem 1.1.6. *The global attractor A is uniformly asymptotically stable.*

Proof. Since A attracts a neighbourhood, it suffices to show that A is stable. Suppose A is not stable. Then from the definition of stability, there exist an open neighbourhood V of A, a sequence of integers $n_j \to \infty$, and a sequence of points $y_j \to y \in A$, as $j \to \infty$ such that $T^k(y_i) \in V$ for all $0 \le k \le n_j$, and $T^{n_j+1}(y_j) \notin V$. Now the sequence $\{y_j\}_{j=1}^{\infty}$ is bounded, thus $T^{n_j}(y_j)$ is contained in a bounded adsorbing set for all sufficiently large j. Without loss of generality we may assume that $T^{n_j}(y_j) \to z$. Note that $z \in A$ since it is an ω-limit point of the set $\{y_j\}_{j=1}^{\infty}$. Hence $Tz \in A$. But this contradicts the fact that $Tz \notin V$, and this contradiction shows that A is stable and thus is uniformly asymptotically stable. □

Remark. For the stability of the global attractor with respect to perturbations of the map T, we refer readers to [1] or [4].

Because of its strong attractivity and strong stability, the global attractor captures the most interesting asymptotic behaviour of the dissipative system. In the next chapter, we shall demonstrate some correlations between the geometric aspects of the global attractor and the asymptotic behaviour of the dissipative system.

§1.2. Autonomous Systems of Ordinary Differential Equations

In this section we define and discuss the continuous dynamical system generated by an autonomous ordinary differential equation in $\mathbf{R}^n$. We shall relate the key concepts of differential equations to those of dynamical systems. In particular, the dissipativeness of a differential equation is shown to be equivalent to the existence of a bounded absorbing set, and in turn can be characterized by the existence of a global attractor. Since the proofs of most of the results in this section are

essentially the same as those of the corresponding results in the section §1.1, they are not included.

Let $D \subset \mathbf{R}^n$ be an open set and $x \mapsto f(x) \in \mathbf{R}^n$ be a continuous function defined in D such that solutions to the autonomous system

$$x' = f(x) \tag{2.1}$$

exist and are uniquely determined by the initial conditions. We denote by $x(t, x_0)$ the solution to (2.1) such that $x(0, x_0) = x_0$.

For each $x \in D$, from the existence and uniqueness assumption, the *positive* (*negative*)*semi-orbit* $\gamma^+(x_0) = \{x(t, x_0) : 0 \leq t < \omega\}$ ($\gamma^-(x_0) = \{x(t, x_0) : -\omega < t \leq 0\}$) through x_0 always exists and is uniquely determined by x_0, where $\omega > 0$ can be ∞ or a finite number. The *orbit* through x_0 is $\gamma(x_0) = \gamma^+(x_0) \cup \gamma^-(x_0)$. For a subset B of D, we use the following notations:

(1) $\gamma^+(B) = \cup_{x \in B} \gamma^+(x)$,

(2) $\gamma^-(B) = \cup_{x \in B} \gamma^-(x)$,

(3) $\gamma(B) = \gamma^+(B) \cup \gamma^-(B)$.

A subset B is said to be *positively invariant* (*negatively invariant*) if $x(t, B) \subset B$ for all $t \geq 0$ (for all $t \leq 0$). It is *invariant* if $x(t, B) = B$ for all $t \in \mathbf{R}$. Equilibria, periodic trajectories and any complete trajectory are some examples of invariant sets. More interesting invariant sets include homoclinic orbits, heteroclinic orbits and heteroclinic cycles.

The most important sources of invariant sets are limit sets. For a subset $B \subset D$, the ω-limit set and α-limit set of B are defined as

$$\omega(B) = \cap_{s \geq 0} \operatorname{cl} \cup_{t \geq s} x(t, B)$$

and

$$\alpha(B) = \cap_{s \geq 0} \operatorname{cl} \cup_{t \geq s} x(-t, B).$$

The following result characterizes the limit sets. Its proof is parallel to that of Proposition 1.1 and can be found in [1] or [4].

Proposition 1.2.1. *Suppose* $x \in D$ *and* $B \subset D$. *Then*

(1) $y \in \omega(x)$ $(\alpha(x))$ *if and only if there exists a sequence of real numbers* $t_n \to \infty$ $(t_n \to -\infty)$ *such that* $x(t_n, x) \to y$ *as* $n \to \infty$.

(2) $y \in \omega(B)$ $(\alpha(B))$ *if and only if there exists a sequence of real numbers* $t_n \to \infty$ $(t_n \to -\infty)$ *and a sequence of points* $x_n \in B$ *such that* $x(t_n, x_n) \to y$ *as* $n \to \infty$.

A subset $A \subset D$ is said to *attract* a subset B if $\mathrm{d}(x(t, B), A) \to 0$ as $t \to \infty$.

The following lemma establishes the invariance of limit sets. For its proof, we refer readers to [1] or [4].

Lemma 1.2.2. *Suppose that* B *is a nonempty subset of* D *such that* $\gamma^+(B)$ $(\gamma^-(B))$ *is bounded. Then* $\omega(B)$ $(\alpha(B))$ *is nonempty compact and invariant. Moreover,* $\omega(B)$ *attracts* B. *If* B *is connected then* $\omega(B)$ $(\alpha(B))$ *is connected.*

Define a mapping $\phi_t : D \to \mathbf{R}^n$ by $\phi_t(x_0) = x(t, x_0)$ for each $t \in \mathbf{R}$ so that $x(t, x_0)$ exists. Then each ϕ_t is a diffeomorphism and the one parameter family of mappings $\{\phi_t\}_{t \in \mathbf{R}}$ satisfies the following semigroup property:

(1) $\phi_{t+s} = \phi_t \circ \phi_s$

(2) $\phi_0 =$ id, the identity mapping in $\mathbf{R}^n$.

$\{\phi_t\}_{t \in \mathbf{R}}$ is called the (*continuous*) dynamical system generated by (2.1). Sometimes it is also called the *flow* generated by (2.1). We see that the semi-orbits of (2.1) can be written as

$$\gamma^+(x) = \{\phi_t(x) \,:\, t \geq 0\},$$

and

$$\gamma^-(x) = \{\phi_t(x) \,:\, t \leq 0\},$$

and that the limit sets can be expressed as

$$\omega(B) = \cap_{s \geq 0} \mathrm{cl} \cup_{t \geq s} \phi_t(B),$$

and

$$\alpha(B) = \cap_{\delta \geq 0} \text{cl} \cup_{t \geq s} \phi_{-t}(B).$$

A subset $B \subset D$ is *positively* (*negatively*) *invariant* with respect to (2.1) if $\phi_t(B) \subset B$ for $t \geq 0$ ($t \leq 0$), and is invariant if $\phi_t(B) = B$ for all $t \in \mathbf{R}$. B is attracted by A with respect to (2.1) if and only if B is attracted by A under ϕ_t, i.e.

$$\mathrm{d}(\phi_t(B), A) \to 0 \qquad \text{as} \quad t \to \infty.$$

The continuous dynamical system $\{\phi_t\}_{t \in \mathbf{R}}$ and its discrete counterpart $\{T^k\}_{k \in \mathbf{N}}$ defined in §1.1, though they appear to be similar, behave quite differently. For example, a periodic trajectory of $\{\phi_t\}_{t \in \mathbf{R}}$ is a simple closed smooth curve in $\mathbf{R}^n$, whereas a periodic orbit of $\{T^k\}_{k \in \mathbf{N}}$ is a finite set of points. However, they are related in many ways.

Suppose $U \subset D$ is a subset such that the solution $x(t, x_0)$ of (2.1) exists for all $t \geq 0$ if $x_0 \in U$. Let $T = \phi_1$ then $T^k = \phi_k$ for each $k \in N$. In this way, the continuous dynamical system $\{\phi_t\}_{t \in \mathbf{R}}$ gives rise to a discrete dynamical system $\{T^k\}_{k \in \mathbf{N}}$. Therefore results obtained for discrete systems can be readily applied to continuous systems generated by differential equations.

A subset $D_0 \subset D$ is said to be *absorbing* with respect to (2.1) if the solution $x(t, x_0)$ exists for all $t \geq 0$ if $x_0 \in D$ and each compact subset $B \subset D$ satisfies that $\phi_t(B) \subset D_0$ for all sufficiently large t. The autonomous system (2.1) is said to be *dissipative* if there exists a bounded absorbing set.

When $D = \mathbf{R}^n$, there is another description of dissipativeness in the literature. (2.1) is said to be *uniformly ultimately bounded* if there exists a constant $b > 0$ and a positive function $t(\rho)$ defined for all $\rho > 0$ such that every solution $x(t, x_0)$ to (2.1) with $|x_0| \leq \rho$ exists for all $t \geq 0$ and satisfies

$$|x(t, x_0)| \leq b \qquad \text{for all} \quad t \geq t(\rho).$$

Notice that the ball $B = \{x \in \mathbf{R}^n \ : \ \|x\| \leq b\}$ is absorbing for (2.1), we can see the two concepts are equivalent in this case.

A *global attractor* of (2.1) is a nonempty compact set $A \subset D$ such that A is invariant and attracts every compact subset of D under ϕ_t. It follows that such a set is necessarily unique. Moreover, it is the maximal compact invariant set in D. From the relation between the discrete dynamical system $\{\phi_t\}_{t\in\mathbf{R}}$ and the discrete dynamical system $\{T^k\}_{k\in\mathbf{N}}$ with $T = \phi_1$, we can deduce that if A is the global attractor of (2.1) then it is the global attractor of the mapping T as defined in §1.1. Using this relation and the corresponding results for discrete dynamical systems in §1.1, we can prove the following results. We omit the proofs since they are very similar to the discrete cases.

Theorem 1.2.3. *Suppose that* (2.1) *is dissipative, let B be a bounded absorbing set. Then $A = \omega(B)$ is the global attractor and A does not depend on the choice of B.*

Theorem 1.2.4. *If there exists a bounded connected absorbing set, then the global attractor is also connected.*

An invariant set $K \subset D$ is said to be *stable* if, for any neighbourhood V of K, there is a neighbourhood W of K such that $\phi_t(W) \subset V$ for all $t \geq 0$). K is *asymptotically stable* if it is stable and attracts every point in a neighbourhood. K is said to be *uniformly asymptotically table* if it is stable and attracts a neighbourhood. As we have remarked in section §1.1, the last two concepts are actually equivalent.

Theorem 1.2.5. *The global attractor A is uniformly asymptotically stable.*

Remarks.

(i) For discussions on the stability of the global attractor with respect to perturbations of the vector field of (2.1), we refer the reader to [1] and [4].

(ii) As in the discrete case, the global attractor of a continuous dynamical system also has the strongest attractivity and stability, it therefore captures the most important asymptotic behaviour of the dynamical system, and of solutions of the differential equation.

§1.3. Periodic Systems of Ordinary Differential Equations

In this section nonlinear ordinary differential equations in $\mathbf{R}^n$ which are periodic in time are considered. We will discuss in detail as to how such a periodic equation can be analyzed via the Poincaré map associated with the equation and through the continuous dynamical system it generates on the product space $\mathbf{S}^1 \times \mathbf{R}^n$.

Suppose that $D \subset \mathbf{R}^n$ is an open set and $(t,x) \mapsto f(t,x) \in \mathbf{R}^n$ is a continuous function defined in $\mathbf{R} \times D$, and is periodic in t with period $\omega > 0$, i.e.

$$f(t+\omega, x) = f(t,x)$$

for all $(t,x) \in \mathbf{R} \times D$. We assume that solutions to the ω-periodic system

$$x' = f(t,x) \tag{3.1}$$

are uniquely determined by the initial conditions and denote by $x(t;t_0,x_0)$ the solution to (3.1) such that $x(t_0;t_0,x_0) = x_0$.

We define a mapping $\mathcal{P} : D \to \mathbf{R}^n$ by

$$\mathcal{P}x_0 = x(\omega; 0, x_0) \qquad x_0 \in D \tag{3.2}$$

From the periodicity of f we know that $\mathcal{P}$ is well defined. The mapping $\mathcal{P}$ is called the *Poincaré map associated with* (3.1). In this way, the periodic system (3.1) gives rise to a discrete dynamical system.

System (3.1) can be studied using the associated Poincaré map $\mathcal{P}$. For example, a fixed point of $\mathcal{P}$ corresponds to an ω-periodic solution of (3.1); a periodic point of $\mathcal{P}$ gives rise to a subhamonic solution of (3.1) and vice versa. Moreover, asymptotic behaviour of the solution $x(t;0,x_0)$ can be well captured by that of $\{\mathcal{P}^n(x_0)\}_{n\in\mathbf{N}}$.

Remark. For each $t_0 \in \mathbf{R}$, we can define in the same way as in (3.2) a Poincaré map

$$\mathcal{P}_1 x_0 = x(\omega; t_0, x_0) \quad \text{for all } x_0 \in D.$$

It is demonstrated in [5] that, for different choices of t_0, the corresponding Poincaré maps are topologically conjugate. In what follows, we always define $\mathcal{P}$ corresponding to $t_0 = 0$.

The periodic system (3.1) can also give rise to a continuous dynamical system on a product space. Rewrite (3.1) as the following autonomous system in $\mathbf{S}^1 \times \mathbf{R}^n$, where $\mathbf{S}^1$ is the unit circle,

$$\begin{aligned} x' &= f(\theta, x) \\ \theta' &= 1 \end{aligned} \tag{3.3}$$

Let ϕ_t be the flow generated by (3.3). Then it is a continuous dynamical system on $\mathbf{S}^1 \times \mathbf{R}^n$. If we denote $t \pmod{\omega}$ by $[[t]]$ for any $t \in \mathbf{R}$, then $[[t]]$ can be identified as a point in $\mathbf{S}^1$. Now for any $([[t_0]], x_0) \in \mathbf{S}^1 \times D$, the flow ϕ_t can be expressed as

$$\phi_t([[t_0]], x_0) = \big([[t_0 + t]], x\big(t + t_0; t_0, x_0\big) \big).$$

Observe that a periodic solution of (3.1) of period commensurate with ω corresponds to a closed orbit of $\{\phi_t\}_{t\in\mathbf{R}}$ on $\mathbf{S}^1 \times D$; a periodic solution to (3.1) of period incommensurate with ω, however, gives rise to a quasiperiodic orbit of $\{\phi_t\}_{t\in\mathbf{N}}$ on $\mathbf{S}^1 \times D$.

The Poincaré map $\mathcal{P}$ can also be derived from the flow $\{\phi_t\}_{t\in\mathbf{R}}$ on $\mathbf{S}^1 \times D$. In fact, if we denote the projection from $\mathbf{S}^1 \times D$ onto the second factor by π and choose a global cross-section Σ_0 of $\mathbf{S}^1 \times D$ by

$$\Sigma_0 = \{\, ([[t]], x) \in \mathbf{S}^1 \times D \,:\, [[t]] = 0 \,\}, \tag{3.4}$$

then Σ_0 is homeomorphic to D, and the Poincaré map defined in (3.2) can be identified with the mapping $\mathcal{P} : \Sigma_0 \to \Sigma_0$ given by $\mathcal{P} = \pi\, \phi_\omega$. In this thesis we will study the ω-periodic system (3.1) using both the associated Poincaré map and the flow $\{\phi_t\}_{t\in\mathbf{R}}$ in $\mathbf{S}^1 \times D$ generated by (3.1), depending on the circumstances.

The ω-periodic System (3.1) is said to be *dissipative* if the associated Poicaré map is dissipative, namely has a bounded absorbing set in $D \subset \mathbf{R}^n$. When $D =$

$\mathbf{R}^n$, (3.1) is said to be *uniformly ultimately bounded* if there exists a constant $b > 0$ and a positive function $h(\rho)$ defined for all $\rho > 0$ such that every solution $x(t; t_0, x_0)$ to (3.1) with $|x_0| \leq \rho$ exists throughout $t_0 \leq t < \infty$ and satisfies

$$|x(t; t_0, x_0)| \leq b \tag{3.5}$$

for all $t > t_0 + h(\rho)$. The following result shows that when $D = \mathbf{R}^n$, the concepts of dissipativeness and uniform ultimate boundedness are equivalent.

Proposition 1.3.1. *When $D = \mathbf{R}^n$, system (3.1) is uniformly ultimately bounded if and only if the associated Poincaré map $\mathcal{P}$ is dissipative, or equivalently, $\mathcal{P}$ has a bounded absorbing set.*

Proof. It is obvious that when (3.5) is satisfied, the bounded subset $\{x \in \mathbf{R}^n : |x| \leq b\}$ is absorbing under $\mathcal{P}$. Conversely, suppose $\mathcal{P}$ has a bounded absorbing set $B \subset \mathbf{R}^n$ such that $|x| \leq b$ for all $x \in B$. Then for each $r > 0$, there exists a $h(r) > 0$ such that $|\mathcal{P}^n(x_0)| \leq b$ for all $n \geq h(r)$ whenever $x_0 \in \mathbf{R}^n$ and $|x_0| \leq r$. This implies that $x(t; 0, x_0)$ satisfies (3.5). For $t_0 \in \mathbf{R}$, using the following property of solutions to (3.1):

$$x(t; t_0, x_0) = x(t; 0, x(0; t_0, x_0)),$$

we can deduce that a general solution $x(t; t_0, x_0)$ also satisfies (3.5). Therefore (3.1) is dissipative. □

From Proposition 1.3.1 and Theorem 1.1.4 we arrive at the following result.

Theorem 1.3.2. *Suppose the system (3.1) is dissipative. Then the associated Poincaré map $\mathcal{P}$ has a global attractor in D.*

§1.4. An Example: the Lorenz Model

In this section, as an illustration, we consider a system of three differential equations proposed by E. N. Lorenz [2] as an indication of the limits of predictability in

weather prediction. The system is a three-mode Galerkin approximation (one in velocity and two in temperature) of the Boussinesq equations for fluid convection in a two-dimensional layer heated from below. The equations are

$$\begin{aligned} x' &= -\sigma x + \sigma y \\ y' &= r x - y - x z \\ z' &= -b z + x y \end{aligned} \tag{4.1}$$

where σ, r, b are three positive parameters. The set of parameter values considered by Lorenz himself are $\sigma = 10$, $r = 20$, $b = 8/3$. There has been a considerable amount of numerical analysis on the Lorenz model for a wide range of parameters. A good reference is [3]. The numerical analysis suggests the existence of a strange global attractor for a certain range of parameters. In this section, we show that (4.1) is dissipative and thus possesses a global attractor. In later chapters, we shall discuss further properties of this attractor.

Theorem 1.4.1. *If* $b > 1$ *the system* (4.1) *is dissipative and the global attractor is contained in the region*

$$D = \{(x, y, z) \in \mathbf{R}^3 : |x| \leq \rho,\ y^2 + (z - r)^2 \leq \rho^2\},$$

where $\rho = \frac{r b}{2\sqrt{b-1}}$.

Proof. Let $V(y, z) = y^2 + (z - r)^2$. Differentiating V along the solutions of (4.1), we have

$$\begin{aligned} V' &= 2 y y' + 2(z - r) z' \\ &= 2 y (r x - y - x z) + 2(z - r)(-b z + x y) \\ &= -2 y^2 - 2 b z^2 + 2 b r z \\ &= -2 V + 2(1 - b)\left(z - \frac{(2 - b) r}{1 - b}\right)^2 + \frac{b^2 r^2}{2(b - 1)}. \end{aligned}$$

Thus $V' < 0$ whenever $V \leq \rho', \rho' > \rho$. Therefore if $(x(t), y(t), z(t))$ is a solution to (4.1) such that $V(y(t_0), z(t_0)) = \rho'$ then $V(y(t), z(t)) < \rho'$ for all

$t > t_0$. Thus $|y(t)| < \rho'$ for all $t \geq t_0$. Now from the first equation of (4.1) we have

$$\begin{aligned}(x^2)' &= 2\,x\,x' = 2\,x\,(-\sigma\,x + \sigma\,y) \\ &\leq 2\,\sigma|x|\,(\rho' - |x|)\end{aligned}$$

when $t \geq t_0$. Hence $|x(t)|$ is strictly decreasing whenever $|x(t)| > \rho'$. This shows that for each $\rho' > \rho$, the bounded set

$$\{\,(x,y,z) \in \mathbf{R}^3 \;:\; |x| \leq \rho',\ y^2 + (r-z)^2 \leq \rho'^2\,\}$$

is absorbing in $\mathbf{R}^3$. Therefore (4.1) is dissipative and the global attractor is contained in $\{\,(x,y,z) \in \mathbf{R}^3 \;:\; |x| \leq \rho,\ y^2 + (r-z)^2 \leq \rho\,\}$. □

§1.5. Bibliography for Chapter I

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CHAPTER II

LOWER ESTIMATES FOR THE HAUSDORFF DIMENSION OF THE GLOBAL ATTRACTOR

As we have seen in Chapter I, for a dissipative system $\{T^k\}_{k\in\mathbf{N}}$, the global attractor captures the most interesting asymptotic behaviour of the system. It is then of great interest to investigate implications of the geometry of the global attractor to the dynamical behaviour of the system. In this chapter, we establish a correlation between the Hausdorff dimension of the global attractor and the existence of certain invariant structures. The definition and properties of the Hausdorff dimension are reviewed in Appendix C.

Our main results in this chapter may be best demonstrated by Theorem 2.2.3 of Section 2.2, which states that if, for some integer $m \geq 0$, the Hausdorff dimension of the global attractor A $(\dim_H A)$ is strictly less than $m+1$, and the domain of T satisfies certain connectivity assumption, then A contains no normal m-boundary. The connectivity required of D is the notion of bounded m-connectedness, which is defined in §2.1. When $m = 0, 1$, this concept agrees with pathwise connectedness and simply connectedness, respectively. Moreover, a convex set is boundedly m-connected for all integer $m \geq 0$. The concept of normal m-boundary is a natural extension to other dimensions of the concept of simple closed curves with finite length. For example, when $m = 0$, this result allows us to conclude that $\dim_H A < 1$ implies that $A = \{\bar{x}\}$ where $\{\bar{x}\}$ is a globally asymptotically stable fixed point. The case $m = 1$ contains a result of R.A. Smith which says that no simple closed piece-wise smooth curves can be invariant under $\{T^k\}_{k\in\mathbf{N}}$ if $\dim_H A < 2$. There is an inaccuracy in Smith's hypotheses, which can be corrected by adding a key assumption that T is 1-1 in A. An example is given to show that such an assumption is necessary.

The significance of this result lies in that it establishes a correlation between the geometry of the global attractor and the dynamical behaviour of the dissipative

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system. It can be useful in two different ways. On the one hand, it gives us a way of obtaining lower estimates of the Hausdorff dimension of the global attractor which is a well-known difficult problem. Suppose that D is convex or there is a convex bounded absorbing set. By detecting the existence in A of a normal m-boundary, invariant or not, we can conclude that $\dim_H A \geq m+1$. The only other approach to this problem we are aware of is given by R. Temam and his colleagues (see [9], Chapter 7). Their method is to estimate the dimension of unstable manifolds of hyperbolic equilibria.

On the other hand, this result, when applied to differential equations, gives a weak condition to preclude existence of some invariant structures such as periodic orbits and higher dimensional invariant tori. The case $m = 1$ will be used in Chapter IV to derive Bendixson's criterion in $\mathbf{R}^n$.

We present our main result (Theorem 2.2.1 and Theorem 2.2.2 in Section 2.2) in the general setting of discrete systems in a Banach space so that it can be used to derive Bendixson's criterion for delay differential equations and partial differential equations. Development in this direction, however, is not in the scope of the present work and will not be discussed.

In Section 2.1, we define the concept of normal m-boundary. Our results for discrete and continuous dynamical systems are discussed in Sections 2.2 and 2.3, respectively. Applications to periodic ordinary differential equations are considered in Section 2.3.

§2.1. Surfaces and Their Boundaries

The purpose of this section is to define the concept of normal m-boundary and other related terminologies which will be used throughout the chapter.

Let $U \subset \mathbf{R}^{m+1}$ be a bounded connected open set whose boundary ∂U is smooth. Let $\overline{U}$ denote the closure of U. Let X be a general normed linear space. A map $\varphi \in C(\overline{U} \to X)$ will be called a $(m+1)$-*surface* in X, and the restriction of φ to ∂U, denoted by $\partial\varphi$, is called the *boundary* of φ. A map

$\psi \in C(\partial U \to X)$ will be called a *m-boundary* in X. A point $x \in \psi(\partial U)$ is said to be *regular* if the tangent space at x has dimension m. A m-boundary ψ is said to be *normal* if it is one to one, Lipschitz continuous on ∂U and has at least one regular point. When $m = 1$, a normal 1-boundary is also called a simple closed *rectifiable curve*. Note that a m-boundary ψ is not necessarily the boundary of a $(m+1)$-surface. It will be said to be *the boundary of* a $(m+1)$-surface φ, denoted by $\partial\varphi \cong \psi$, if $\partial\varphi(\partial U) = \psi(\partial U) =: \Gamma$, and there exists a continuous one to one mapping $S : \Gamma \to \Gamma$ such that $\partial\varphi = S\psi$. This is also described as ψ *bounds* a $(m+1)$-surface φ. We can see from the definition that if φ is a $(m+1)$-surface in X and $F : X \to X$ is a continuous mapping then $F\varphi$ is also a $(m+1)$-surface in X and its boundary satisfies $\partial(F\varphi) = F\partial\varphi$.

The *trace* of a $(m+1)$-surface $\varphi \in C(\overline{U} \to X)$ is defined to be the subset $\varphi(\overline{U})$ of X, and the trace of a m-boundary $\psi \in C(\partial U \to X)$ is $\psi(\partial U)$. If the trace of φ is contained in a subset B of X, we say that φ is a $(m+1)$-surface in B, or that B contains a $(m+1)$-surface φ. The same terminology also applies to a m-boundary ψ. The term ψ bounds a $(m+1)$-surface φ in B should be understood as $\varphi(\overline{U}) \subset B$ and $\partial\varphi \cong \psi$. As an example, a 0-boundary can have as trace a pair of points or a single point. It is normal only if its trace is a pair of distinct points. The trace of a 1-surface is a usual curve, and the trace of a 1-boundary can either be a single closed curve or consist of a family of closed curves with some of the curves possibly degenerate to points. It is normal only if all the member curves of its trace are simple closed rectifiable curves. In particular, the usual simple closed piece-wise smooth curves can be regarded as the traces of normal 1-boundary by our definition (See Figure 2.1.1).

Whether a m-boundary can bound within an open set $D \subset X$ a $(m+1)$-surface is closely related to the connectivity of D. One can easily see that D is path connected if and only if every 0-boundary bounds (is the boundary of) a 1-surface in D. The simply connectedness of D can be characterized by the property that every 1-boundary bounds a 2-surface in D. We say D is *compactly* (*boundedly*) *m-connected* if, for each compact subset K of D and any

family of m-boundaries $\{\psi_\alpha\}_{\alpha\in\Lambda}$ in K, there exists a compact (bounded) set B such that

$$K \subset B \subset D$$

and each ψ_α bounds a $(m+1)$-surface in B. Obviously compact m-connectedness implies bounded m-connectedness, and in a finite dimensional space the two are equivalent. Moreover, if D is a convex subset of a Banach space X , then it is compactly m-connected for all integers m. To see this, notice that for a compact set $K \subset D$, the closed convex hull $\overline{co}(K)$ of K is also a compact subset of D. Then a $(m+1)$-surface in D can be constructed as a cone on each m-boundary.

Another example is shown in Figure 2.1.2, where D is the annular region. The simple closed curve C shown in Figure 2.1.2(a) can be regarded as the trace of a normal 1-boundary which can not bound any 2-surface in D, whereas the pair of closed curves in Figure 2.1.2(b) is the trace of a 1-boundary which is the *boundary* of a 2-surface whose trace is the region between these two closed curves.

Remark. The following discussions involve the Frechét differentiability of a normal $m-$boundary in X.

(i). Since X is a general normed linear space, Frechét differentiability (almost everywhere in $\mathbf{R}^m$) of a Lipschitz continuous mapping $\psi : \mathbf{R}^m \to X$ can no longer be implied by the Rademacher theorem (see [5]) as is in the case when X is of finite dimension. In fact, examples exist to show that a Lipschitz continuous mapping from $\mathbf{R}$ to $\mathrm{L}^1(0,1)$ may fail to be Frechét differentiable anywhere (see [10]). It is a result of R. R. Phelps ([7]) which shows that every Lipschitz continuous mapping $\psi : \mathbf{R}^m \to X$ is Gateaux differentiable almost everywhere provided X is a Banach space and satisfies the so-called 'Radon-Nikodym property'. A Banach space X is said to satisfy the Radon-Nikodym property if every Lipschitz continuous mapping $\psi : \mathbf{R} \to X$ is Frechét differentiable almost everywhere. Earlier results of N. Dunford and B. J. Pettis [3] showed that such spaces include Banach spaces which are reflexive. For example the L^p spaces, when $1 < p < \infty$, have the Radon-Nikodym property. Since, for a mapping $\psi : \mathbf{R}^m \to X$, Frechét differentiability

is equivalent to Gateaux differentiability, Phelps' result implies that ψ is Frechét differentiable almost everywhere when X is a Banach space satisfying the Radon-Nikodym property.

(ii). We assume that a normal m-boundary ψ has at least one regular point; namely, there exists at least a $x \in \psi(\partial U)$ such that the tangent space at x has dimension m. In the case when X is of finite dimension, this assumption is implied by the lipschitzian continuity, because ψ is then rectifiable and thus has many regular points. In fact, a version of Sard's Theorem holds for such mappings (see [5]).

We will assume throughout this section that X is a Banach space which satisfies the Radon-Nikodym property.

The following result which establishes a correlation between the existence of a certain type of m-boundary in a compact set K and the Hausdorff dimension of K is crucial to the development in this chapter.

Proposition 2.1.1. *Assume that K is a compact subset of X and that there is a normal m-boundary ψ in K. Suppose, for each neighbourhood V of K, ψ bounds a $(m+1)$-surface in V. Then* $\dim_H K \geq (m+1)$.

The case $m = 0$ of this proposition may be proved as follows. Suppose ψ is a normal 0-boundary in K. Then its trace is a pair of distinct points x_1 and x_2 in K. Let $\delta > 0$ and $\{B_i\}_i$ be a cover of K by open sets of diameter $|B_i| \leq \delta$. Such an open cover is called a δ-cover of K (see Appendix C). Now $V = \cup_i B_i$ is an open neighbourhood of K. From the assumptions of the proposition we know that ψ bounds a 1-surface φ whose trace γ is a continuous curve in V connecting x_1 and x_2 (see Figure 2.1.3). Interpolating γ in V by a polygonal curve if necessary, we may assume that γ is rectifiable. Obviously, $\sum_i |B_i|$ is greater than the length of γ which is bounded below by the distance d between x_1 and x_2. Therefore we arrive at

$$0 < d \leq \sum_i |B_i|$$

for all δ-covers $\{B_i\}_i$ of K. This implies that the 1-dimensional Hausdorff measure of K satisfies $\mathcal{H}^1(K) > 0$, and thus $\dim_H K \geq 1$ by the definition of Hausdorff dimension (see Appendix C).

Before we proceed to prove the proposition for general m, some preparations are needed. Suppose $\psi \in C(\partial U \to X)$ is a normal m-boundary. Then, from the remark preceding Proposition 2.1.1, we know ψ is Frechét differentiable almost everywhere and it has at least one regular point. Thus its trace $\Gamma = \psi(\partial U)$ has a unique tangent space almost everywhere and the tangent space is of dimension m at least at one point. Without loss of generality, we may assume that Γ contains 0 and its tangent space at 0 is a m-dimensional subspace X_1 of X. Let X_2 be a closed subspace of X complementary to X_1. Then

$$X = X_1 \oplus X_2$$

and there exists a constant $c > 0$ such that

$$c(\,|x_1| + |x_2|\,) \leq |x_1 + x_2| \leq |x_1| + |x_2|$$

for any $x_i \in X_i$, $i = 1, 2$ (see [2]).

We now define a rotation mapping $\mathcal{R} : X \to \mathbf{R}^{m+1}$ by

$$\mathcal{R}(x) = (x_1, |x_2|) \qquad x \in X, \tag{1.1}$$

where $x = x_1 + x_2$, $x_i \in X_i$, $i = 1, 2$, and the vector x_1 is identified with its coordinate vector with respect to a fixed basis in X_1. We will assume that $\mathbf{R}^{m+1}$ is endowed with the euclidean norm $\|\cdot\|$. Then there are two norms on X_1, the norm $|\cdot|$ inherited from X and the euclidean norm $\|\cdot\|$ of $\mathbf{R}^m$, and there are constants c_1, c_2 such that

$$c_1|x_1| \leq \|x_1\| \leq c_2|x_1| \qquad \text{for all } x_1 \in X_1.$$

The following lemma gives the properties of the mapping $\mathcal{R}$.

Lemma 2.1.2 . *There are constant* $K_1, K_2 > 0$ *such that*

(1) $K_1|x| \leq \|\mathcal{R}(x)\| \leq K_2|x|$ *for all* $x \in X$,

(2) $\mathcal{R}$ *is Lipschitz continuous with the Lipschitz constant* $\mathrm{Lip}\mathcal{R} \leq K_2$,

(3) $\mathcal{R}|_{X_1}$ *is a linear isomorphism from* X_1 *onto the subspace of* $\mathbf{R}^{m+1}$

$$\{(y_1, ..., y_{m+1}) \in \mathbf{R}^{m+1} \ : \ y_{m+1} = 0\}.$$

Proof. (3) follows directly from the definition of $\mathcal{R}$. To see that (1) is true, let $x \in X$, and $x = x_1 + x_2$, $x_i \in X_i$, $i = 1,2$. Notice that

$$\|\mathcal{R}(x)\| = \{\|x_1\|^2 + |x_2|^2\}^{\frac{1}{2}}$$

and

$$\begin{aligned}\{\|x_1\|^2 + |x_2|^2\}^{\frac{1}{2}} &\leq \|x_1\| + |x_2| \\ &\leq c_3(|x_1| + |x_2|) \leq \frac{c_3}{c}|x|\end{aligned}$$

with $c_3 = \max\{1, c_2\}$. On the other hand,

$$\begin{aligned}\{\|x_1\|^2 + |x_2|^2\}^{\frac{1}{2}} &\geq (\|x_1\| + |x_2|)/\sqrt{2} \\ &\geq c_4(|x_1| + |x_2|)/\sqrt{2} \geq c_4|x|/\sqrt{2},\end{aligned}$$

with $c_4 = \min\{1, c_1\}$. Hence (1) is proved with $K_1 = c_4/\sqrt{2}$ and $K_2 = c_3/c$. Now (1) and an obvious inequality

$$\|\mathcal{R}(x) - \mathcal{R}(y)\| \leq \|\mathcal{R}(x-y)\|$$

imply (2) $\square$.

Our next lemma proves that there is a cone in X with its vertex at 0, which is symmetric with respect to the tangent space X_1 of Γ at 0, and contains a small portion of Γ around 0 (see Figure 2.1.5)

Let

$$B_r(0) = \{x \in X \ : \ |x| \leq r\}$$

be the ball of radius r in X centred at 0.

Lemma 1.2. *For sufficiently small* $r > 0$, *a ball* $B_r(0)$ *can be found so that* $B_r(0) \cap \Gamma$ *is contained in the cone*

$$C = \{x \in X \ : \ |x_2| \leq \alpha |x_1|\},$$

where the constant α *depends only on* Γ *and* r.

Proof. Let $u_0 \in \partial U$ be such that $\psi(u_0) = 0$, ψ is differentiable at u_0 and the tangent map L of ψ at u_0 is of maximal rank m. Then, for $u \in \partial U$ sufficiently close to u_0,

$$\psi(u) = L(u - u_0) + h(u - u_0)$$

where h satisfies $|h(u - u_0)| = o(\|u - u_0\|)$ for u sufficiently close to u_0. Write $\psi = (\psi_1, \psi_2)$, $h = (h_1, h_2)$ in the coordinates for $X = X_1 \oplus X_2$. Then

$$\psi_1(u) = L(u - u_0) + h_1(u - u_0)$$

$$\psi_2(u) = h_2(u - u_0).$$

Since L is of maximal rank, there is a constant $c > 0$, such that

$$|L(u - u_0)| \geq c\|u - u_0\| \qquad \text{for all} \quad u.$$

and thus there exists a sufficiently small $\epsilon > 0$ such that

$$|\psi_1(u)| \geq (c - \epsilon)\|u - u_0\|$$

$$|\psi_2(u)| \leq \epsilon\|u - u_0\|.$$

for u_0 sufficiently close to u. Therefore

$$|\psi_2(u)| \leq \alpha|\psi_1(u)|$$

when u is sufficiently close to u_0. The ball $B_r(0)$ can be chosen by the continuity of ψ $\square$.

Proof of Proposition 2.1.1. The proposition will be proved by showing the existence of a positive constant ε_0 such that

$$\sum_i |B_i|^{m+1} \geq \varepsilon_0 > 0$$

for all δ-covers $\{B_i\}_i$ of K and all $\delta > 0$. Now $V = \cup_i B_i$ is an open neighbourhood of K. Let $\varphi \in C(U \to V)$ be a $(m+1)$-surface bounded by ψ in V. Then $\varphi(\overline{U}) \subset V$, $\partial\varphi(\partial U) = \psi(\partial U)$ and $\partial\varphi$ is a one to one mapping from ∂U to $\partial\varphi(\partial U)$. Let the subspaces X_1, X_2 of X and the mapping $\mathcal{R}$ be those associated with ψ by Lemma 2.1.2. Then

$$\mathcal{R}\varphi(\overline{U}) \subset \mathcal{R}(V) \subset \cup_i \mathcal{R}(B_i). \tag{1.2}$$

Hence

$$\mathcal{H}^{m+1}(\mathcal{R}\varphi(\overline{U})) \leq \sum_i \mathcal{H}^{m+1}(\mathcal{R}(B_i)) \leq \sum_i |\mathcal{R}(B_i)|^{m+1} \leq K_2^{m+1} \sum_i |B_i|^{m+1}, \tag{1.3}$$

where $\mathcal{H}^{m+1}$ is the Hausdorff measure in dimension $m+1$.

The next part of the proof is to show that

$$\mathcal{H}^{m+1}(\mathcal{R}\varphi(\overline{U})) \geq \epsilon_0 > 0 \tag{1.4}$$

for some ϵ_0 independent of the cover $\{B_i\}$ and the $(m+1)$-surface φ. Then (1.3) and (1.4) will imply $\sum |B_i|^{m+1} \geq \epsilon_0 > 0$, completing the proof.

Let the cone $C = \{x \in X \ : \ |x_2| \leq \alpha |x_1|\}$ and the ball $B_r(0)$ be associated with $\psi(\partial U)$ by Lemma 1.2. Let $\Gamma' = \Gamma \cap B_r(0)$ be the portion of Γ contained in the cone C. Apply the rotation $\mathcal{R}$ to this configuration. Working in $\mathbf{R}^{m+1}$, the range of $\mathcal{R}$, we can see that the image of C under $\mathcal{R}$ is a cone $\widetilde{C}$ in $\mathbf{R}_+^{m+1} = \mathbf{R}^{m+1} \cap \{y_{m+1} \geq 0\}$, given by

$$\widetilde{C} = \{y \in \mathbf{R}_+^{m+1} \ : \ \|(y_1, \cdots, y_m)\| \leq \alpha_1 |y_{m+1}|\}$$

where α_1 is a constant determined by α representing the cone C and by the norm $|\cdot|$ of X. Here and in the following, we use the tilde over a set to indicate

that it belongs to $\mathbf{R}^{m+1}$. Since Γ' is a compact portion of Γ, taking into account that ψ is one to one and $\mathcal{R}$ is bi-Lipschitz, a small ball $\widetilde{B}_s(0)$ in $\mathbf{R}^{m+1}$ with $s < r$ can be chosen so that in $\mathbf{R}^{m+1}$

$$\mathcal{R}(\Gamma) \cap \widetilde{B}_s(0) = \mathcal{R}(\Gamma') \cap \widetilde{B}_s(0).$$

Namely, the part of $\mathcal{R}(\psi)$ intersecting $\widetilde{B}_s(0)$ is contained in the cone $\widetilde{C}$. Now the boundary of the cone $\widetilde{C}$ splits the ball $\widetilde{B}_s(0)$ into three parts: the half contained in $\mathbf{R}^{m+1}_- = \mathbf{R}^{m+1} \cap \{y_{m+1} \leq 0\}$, denoted by $\widetilde{D}_1$; the part contained in $\widetilde{C}$; and the remaining part, denoted by $\widetilde{D}_2$, which is contained in $\mathbf{R}^{m+1}_+$ and does not intersect $\widetilde{C}$ (see Figure 2.1.6) The volume of $\widetilde{D}_2$ is a fraction of that of $\widetilde{B}_s(0)$, and thus

$$\mathcal{H}^{m+1}(\widetilde{D}_2) = \alpha_2 \mathcal{H}^{m+1}(\widetilde{B}_s(0)) = \alpha_2 s^{m+1} \tag{1.5}$$

where $\alpha_2 > 0$ is a constant determined by α_1 and m. Now we claim $\widetilde{D}_2$ is covered by $\mathcal{R}\varphi(\overline{U})$:

$$\widetilde{D}_2 \subset \mathcal{R}\varphi(\overline{U}). \tag{1.6}$$

This and (1.5) will imply

$$0 < \alpha_2 s^{m+1} = \mathcal{H}^{m+1}(\widetilde{D}_2) \leq \mathcal{H}^{m+1}(\mathcal{R}\varphi(\overline{U})),$$

proving the theorem.

To show (1.6), we use the topological degree theory. Take any point p in $\widetilde{D}_2$. The intersection number with $\mathcal{R}(\Gamma) = \mathcal{R}\varphi(\overline{U})$ of any ray emitting from p and passing through 0 is either 1 or -1. This is because φ is one to one on ∂U and $\mathcal{R}$ is one to one at 0. Therefore, for the mapping $\mathcal{R}\varphi : U \subset \mathbf{R}^{m+1} \to \mathbf{R}^{m+1}$, $\deg(\mathcal{R}\varphi, \overline{U}, \widetilde{D}_2) = \deg(\mathcal{R}\varphi, \overline{U}, \widetilde{D}_1) \pm 1$. Now $\deg(\mathcal{R}\varphi, \overline{U}, \widetilde{D}_1) = 0$ since $\mathcal{R}\varphi(\overline{U})$ is totally contained in $\mathbf{R}^{m+1}_+$. This implies $\deg(\mathcal{R}\varphi, \overline{U}, \widetilde{D}_2) \neq 0$. Therefore, from the degree theory, $\widetilde{D}_2 \subset \mathcal{R}\varphi(\overline{U})$. □

§2.2. Discrete Systems in a Banach Space

R.A. Smith proved in [8] that for a continuous mapping $T : \mathbf{R}^n \to \mathbf{R}^n$ which is dissipative with a global attractor A, if $\dim_H A < 2$ then no simple closed piecewise smooth curve can be invariant under T. He also applied this result to derive Bendixson's criterion for autonomous ordinary differential equations in $\mathbf{R}^n$. In this section, we will generalize Smith's result and show that if $\dim_H A < m+1$ for some integer $m \geq 0$, then A contains no normal m-boundary. Our results are proved in a more general setting of discrete dynamical systems in a Banach space so that they may be applied to derive Bendixson's criterion for differential equations with delays and for partial differential equations.

Let X be a Banach space and $D \subset X$ an open subset. Let $T : D \to X$ be a continuous mapping such that $T(D) \subset D$.

A m-boundary $\psi \in C(\partial U \to D)$ is said to be *invariant* under T if $\Gamma = \psi(\partial U)$ is invariant under T.

Theorem 2.2.1. *Assume that a compact set $K \subset D$ attracts compact sets in D under T. If $\dim_H K < m+1$, then there can be no normal m-boundary ψ in K satisfying the following:*

(1) *ψ is invariant under T,*

(2) *T is one to one on $\Gamma = \psi(\partial U)$,*

(3) *ψ bounds a $(m+1)$-surface in D.*

Proof. We prove the theorem by contradiction. Suppose that there is in K a normal m-boundary ψ satisfying (1), (2) and (3) in the theorem, we will show that $\dim_H K \geq m+1$. Let $\varphi \in C(\overline{U} \to D)$ be a $(m+1)$-surface bounded by ψ. Then for each neighbourhood V of K, there exists a $N > 0$ such that

$$T^n(\varphi(\overline{U})) \subset V \qquad \text{for all } n \geq N.$$

Hence $T^N\varphi \in C(\overline{U} \to V)$. Moreover, $\partial(T^N\varphi)(\partial U) = T^N\partial\varphi(\partial U) = T^N(\psi(\partial U)) = \psi(\partial U)$ since ψ is invariant under T. Therefore V contains a $(m+1)$-surface which has ψ as the boundary. The Proposition 2.1.1 in §2.1 then implies that $\dim_H K \geq m+1$. □

Theorem 2.2.2. *Assume that D is compactly (boundedly) m-connected. Suppose that there exists a compact invariant subset K of D which attracts compact (bounded) sets in D, and that T is one to one on K. If $\dim_H K < m+1$, then there can be no normal m-boundary in K.*

Proof. Suppose ψ is a normal m-boundary in K. If ψ is invariant under T, then Theorem 2.1 implies $\dim_H K \geq m+1$. If ψ is not invariant under T. Then $\{T^n\psi\}_{n=-\infty}^{+\infty}$ is a family of m-boundaries in K. By our assumption in D, there exists a compact (bounded) subset B of D such that

$$K \subset B \subset D$$

and each $T^n\psi$ bounds a $(m+1)$-surface φ_n in B. Since K attracts B, there exists, for each neighbourhood V of K, a $N > 0$ such that

$$T^n(B) \subset V \qquad \text{for all } n \leq N.$$

In particular, $T^N(\varphi_{-N}(\overline{U})) \subset V$, and thus $T^N\varphi_{-N}$ is a $(m+1)$-surface in V. Moreover, $\partial(T^N\varphi_{-N})(\partial U) = T^N\partial\varphi_{-N}(\partial U) = T^N T^{-N}(\psi(\partial U)) = \psi(\partial U)$, and thus $T^N\varphi_{-N}$ is a $(m+1)-$ surface which has ψ as the boundary. The Proposition 2.1.1 of §2.1 then implies $\dim_H K \geq m+1$. This, however, contradicts the assumption that $\dim_H K < m+1$, and therfore the theorem is proved. □

Remark. Since, when X is of infinite dimension, bounded subsets of X may not be compact. The attractivity assumption on the compact invariant set K in Theorem 2.2.2 is weaker than that required for a global attractor. As a matter of fact, when a compact invariant set K exists which attracts compact sets, the mapping T is called by Hale (see [6]) 'compact dissipative', which is different from T being dissipative. Some other notions of dissipativity for dynamical systems in infinite dimensional spaces as well as related examples can also be found in [6].

Next, we assume that the mapping T is dissipative and denote the global attractor in D by A. Then A is the maximal compact invariant subset of D

and attracts bounded subsets of D. From Theorem 2.2.1 and Theorem 2.2.2 we can derive the following results.

Theorem 2.2.3. *Suppose* $\dim_H A < m+1$ *for some integer* $m \geq 0$. *Then there can be no normal* m*-boundary* ψ *in* A *satisfying the following:*

(1) ψ *is invariant under* T,

(2) T *is one to one on* $\Gamma = \psi(\partial U)$,

(3) ψ *bounds a* $(m+1)$*-surface in* D.

Theorem 2.2.4. *Suppose that* D *is boundedly* m*-connected and* T *is one to one on the global attractor* A. *If* $\dim_H A < m+1$, *then there can be no normal* m*-boundary in* A.

When $m = 1$, Theorem 2.2.3 yields the result of R.A. Smith.

Theorem 2.2.5. *Suppose* D *is simply connected and* A *is the global attractor of* T *in* D. *If* $\dim_H A < 2$, *then* A *contains no invariant simple closed piecewise smooth curve on which* T *is one to one.*

Remark. In his result (Theorem 5, [8]), Smith does not assume that T is one to one, and the result as stated is false as the following example demonstrates. In fact, in his proof there is a tacit assumption that T is one to one, as stated in Theorem 2.2.5.

Example. Let $\mathbf{C}$ be the complex plane. We consider a mapping $T : \mathbf{C} \to \mathbf{C}$ which will be defined in three steps. More precisely, we will define T as $T = T_3 \circ T_2 \circ T_1$, where the mappings T_1, T_2, T_3 are as follows:

(i) T_1 contracts $\mathbf{C}$ onto the closed unit disc $\overline{U} \subset \mathbf{C}$ and leaves $\overline{U}$ unchanged:

$$T_1 z = \begin{cases} \frac{z}{|z|}, & \text{if } |z| > 1 \\ z, & \text{if } |z| \leq 1. \end{cases}$$

(ii) T_2 squeezes $\overline{U}$ onto the half of the unit circle $C = \partial U$ contained in the region $\mathrm{Re} z \geq 0$:

$$T_2 z = \sqrt{1-y^2} + iy, \quad \text{if } z = x + iy \quad \text{and } x^2 + y^2 \leq 1.$$

(iii) T_3 doubles the argument of each point on C :

$$T_3 z = e^{i2\theta} \qquad \text{if} \quad z = e^{i\theta}, \ \theta \in [0, 2\pi].$$

The mapping T is obviously continuous on $\mathbf{C}$, and is dissipative because of T_1. It is also easy to check that the unit circle C is invariant and attracts compact subsets of $\mathbf{C}$ under T. Obviously $\dim_H C = 1$. The mapping T evades Theorem 2.2.2 since T is not 1-1 on C.

The significance of Theorem 2.2.3 and Theorem 2.2.4 is two-fold.

(I). They provide a way of obtaining lower estimates of the Hausdorff dimension of the global attractor. The following result is in this spirit.

Theorem 2.2.7. *Assume that D is convex and T is one to one on the global attractor A. Suppose A contains a normal m-boundary. Then $\dim_H A \geq m+1$.*

Proof. Since convex set is boundedly m-connected, the theorem follows from Theorem 2.2.2. □

When $m = 0, 1$, this yields the following results.

Corollary 2.2.8. *Suppose that D is path connected and T in one to one on the global attractor A. If A contains more than one point, then $\dim_H A \geq 1$.*

Corollary 2.2.9. *Suppose that D is simply connected and T in one to one on the global attractor A. If A contains a simple closed rectifiable curve, then $\dim_H A \geq 2$.*

The assumptions on the absorbing set in Corollary 2.2.8 and Corollary 2.2.9 imply that D is boundedly 0-connected and 1-connected, respectively. Thus both results follow from Theorem 2.2.4.

It is a well-known fact that, for a dynamical system $\{T^k\}_{k\in\mathbf{N}}$ in an infinite dimensional space, it is not realistic to assume T is a one to one mapping on the whole space. For example, a differential delay equation $x' = f(x_t)$ can generate a discrete dynamical system $\{T^k\}_{k\in\mathbf{N}}$ in the space of all continuous functions on [-1,0), and usually T is not one to one because of the lack of backwards uniqueness of the solutions of the delay equation. However, as Hale demonstrates in [6], it is very often that T is one to one on the global attractor. In the case of the delay equations, it is proved in [6] Chapter IV that T is one to one on the global attractor when f is analytic. This shows that the assumption that T is one to one on the attractor is reasonably realistic.

Estimating $\dim_H A$ from below is well-known to be a hard problem. The only other approach known to us is provided by R. Temam, C. Foias and A. Eden (see [9] Chapter 7). By establishing that the unstable manifolds of equilibria are contained in the global attractor and estimating the dimension of the unstable manifolds at hyperbolic equilibria, they obtain a lower bound for $\dim_H A$. It is important to note that neither of the two approaches here can provide a nontrivial fractional lower bound to $\dim_H A$.

(II). Theorem 2.2.3 and Theorem 2.2.4 also provide a sufficient condition to rule out the existence of certain invariant structures. The following result is a typical example.

Corollary 2.2.10. *Suppose that D is simply connected and the global attractor A satisfies $\dim_H A < 2$. Then no simple closed rectifiable curve in D can be invariant under T.*

§2.3. Ordinary Differential Equations in $\mathbf{R}^n$

We have seen in Chapter I that a periodic (including autonomous) ordinary differential equation in $\mathbf{R}^n$ can generate a discrete dynamical system in $\mathbf{R}^n$. In the case of autonomous equations, this is given by $\{T^k\}_{k\in\mathbf{N}}$ with $T = \phi_1$, where $\{\phi_t\}_{t\in\mathbf{R}}$

is the flow generated by the differential equation, while for an ω-periodic equation, this is given by $\{\mathcal{P}^k\}_{k\in\mathbf{N}}$, where $\mathcal{P}$ is the associated Poincaré map. Since both T and $\mathcal{P}$ are diffeomorphisms, results in §2.1 can be readily applied to both cases. This is the subject of the present section. In the subsection §2.3.1, autonomous equations are treated. In §2.3.2, ω-periodic equations are discussed. In both cases, much attention is being paid to implications to the asymptotic behaviour of solutions.

§2.3.1. Autonomous Systems

Consider an autonomous system in $\mathbf{R}^n$

$$x' = f(x) \tag{3.1}$$

where the function $x \mapsto f(x) \in \mathbf{R}^n$ is C^1 in an open subset D of $\mathbf{R}^n$. Let φ_t be the flow generated by (3.1). Assume that (3.1) is dissipative with the bounded absorbing set $D_0 \subset D$. Then we know from Chapter I that $A = \omega(D_0)$ is the global attractor which is the maximal compact invariant set in D and attracts every compact subset of D under ϕ_t.

Apply Theorem 2.2.3 and Theorem 2.2.4 to the diffeomorphism ϕ_t for any fixed t we have the following results.

Theorem 2.3.1. *If* $\dim_H A < m+1$, *then there can be no normal* m*-boundary in* A *which bounds a* $(m+1)$*-surface in* D.

Theorem 2.3.2. *Suppose that* D *is boundedly* m*-connected. If* $\dim_H A < m+1$, *then there can be no normal* m*-boundary in* A.

This gives a lower estimate of $\dim_H A$.

Theorem 2.3.3. *Suppose that the absorbing set is convex. If* A *contains a normal* m*-boundary, then* $\dim_H A \geq m+1$.

Remark. The m-boundary in Theorem 2.3.3 need not be invariant.

When $m = 1$, Theorem 2.3.1 gives us a weak criterion for the nonexistence of simple closed rectifiable curves which are invariant with respect to system (1.1)

Theorem 2.3.4. *Suppose D is simply connected. If $\dim_H A < 2$, then no simple closed rectifiable curve in D can be invariant with respect to* (3.1).

Remarks.

(i) Since a periodic trajectory give rise to a simple closed curve invariant with respect to (3.1), under the condition $\dim_H A < 2$, Theorem 2.3.4 tells us that (3.1) can not have periodic trajectories.

(ii) Theorem 2.3.4 was first obtained by R. Smith [8]. It provides the weakest condition so far for a dissipative system not to possesse periodic trajectories.

(iii) Theorem 2.3.4 can be used to derive higher dimensional Bendixson's criteria for dissipative systems (3.1). We will see in Chapter IV that, using the upper estimates on $\dim_H A$ given in the Chapter III, concrete criteria can be derived from Theorem 2.3.4. which not only preclude periodic solutions but also trajectories of the following types:

(1) a homoclinic trajectory;

(2) a pair of heteroclinic trajectories of the same equilibria;

(3) a heteroclinic cycle.

§2.3.2. Periodic Systems

Consider a periodic system in $\mathbf{R}^n$

$$x' = f(t, x) \tag{3.2}$$

where the function $(t, x) \mapsto f(t, x) \in \mathbf{R}^n$ is C^1 in $\mathbf{R} \times D$ for some open subset D of $\mathbf{R}^n$ and is ω-periodic in t. Let $\mathcal{P}$ be the Poincaré map associated with (3.2). Assume that (3.2) is dissipative and $D_0 \subset D$ is a bounded absorbing set.

Then, from Chapter I, we know $A = \omega(D_0)$ is the global attractor for $\mathcal{P}$ and A is the maximal compact invariant set in D and attracts bounded subsets of D under $\mathcal{P}$.

Applying Theorem 2.2.3 and Theorem 2.2.4 to the Poincaré map $\mathcal{P}$ and taking into account that $\mathcal{P}$ is a diffeomorphism, we have the following results.

Theorem 2.3.5. *If* $\dim_H A < m + 1$, *then there can be no normal* m*-boundary in* A *which bounds a* $(m + 1)$*-surface in* D.

Theorem 2.3.6. *Suppose that* D *is boundedly* m*-connected. If* $\dim_H A < m+1$, *then there can be no normal* m*-boundary in* A.

In particular, when $m = 0, 1$, we have the following results.

Theorem 2.3.7. *Suppose that* D *is pathwise connected. If* $\dim_H A < 1$, *then* $A = \{x_0\}$ *where* x_0 *is a globally asymptotically stable fixed point of* $\mathcal{P}$ *in* D.

Proof. The assumption on D implies that D is boundedly 0-connected. Therefore by Theorem 2.2.4, A is a single point x_0 if $\dim_H A < 1$. From the maximal invariance of A we know x_0 is the only fixed point of $\mathcal{P}$ and the global stability of x_0 follows from that of the global attractor given in Theorem 1.1.6 in Chapter I.

Theorem 2.3.8. *Suppose* D *is simply connected. If* $\dim_H A < 2$, *then* A *contains no simple closed rectifiable curves.*

Theorem 2.3.7 has the following implication to the asymptotic behaviour of solutions to (3.2).

Theorem 2.3.9. *Suppose that (3.2) is dissipative and* D *is path-wise connected. If* $\dim_H A < 1$, *then (3.2) has a unique* ω*-periodic solution which is globally uniformly asymptotically stable.*

Proof. Recall that a fixed point x_0 of the Poincaré map $\mathcal{P}$ corresponds to a ω-periodic solution to (3.2), and the stability of x_0 under $\mathcal{P}$ is equivalent to the uniform stability of the corresponding ω-periodic solution. Therefore, Theorem 2.3.9 follows from Theorem 2.3.7.

Theorem 2.3.8 also has strong implications to the asymptotic behaviour of solution to (3.2), which will be discussed in Chapter IV, where we will see that under the conditions of Theorem 2.3.8, (3.2) can not have quasi-periodic solutions.

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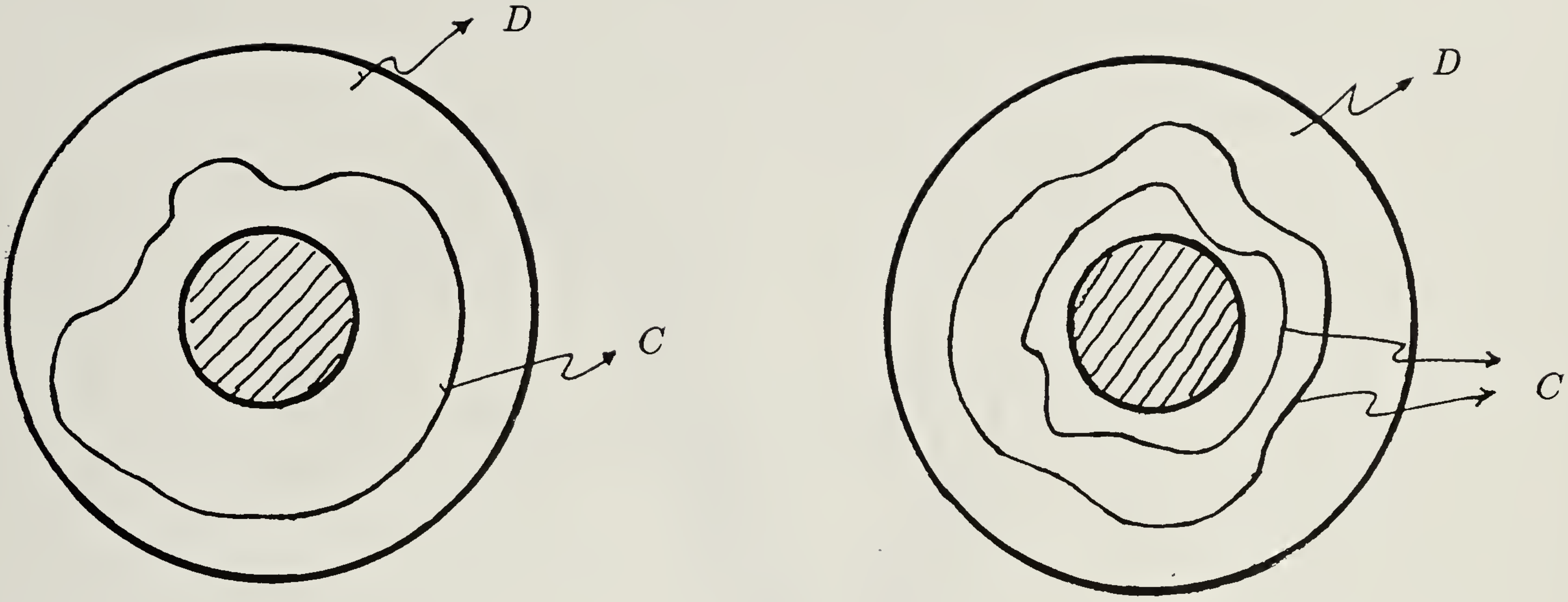


(a) A simple closed curve C.

(b) A pair of simple closed curves.

Figure 2.1.2

Figure 2.1.1 A normal 1-boundary

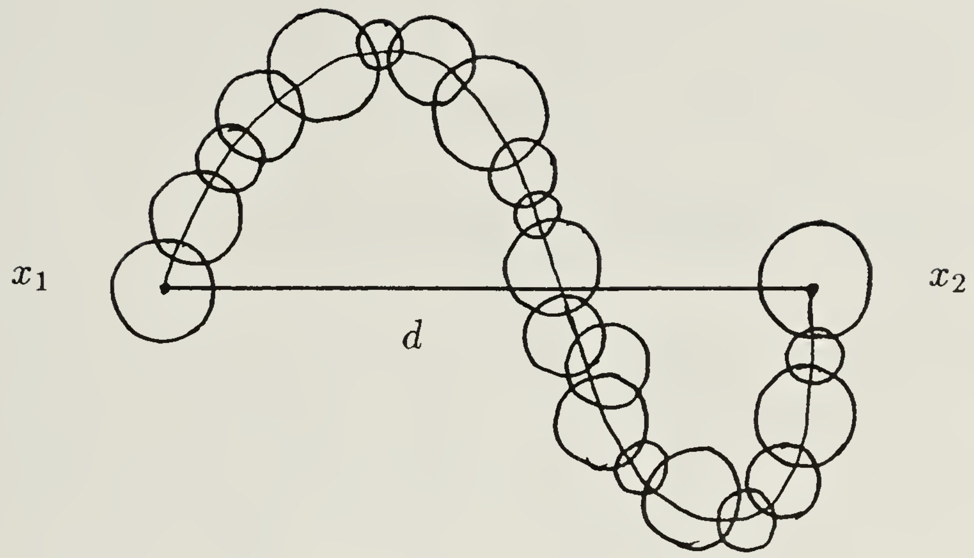


Figure 2.1.2. Figure 2.1.3. Case $m = 0$.

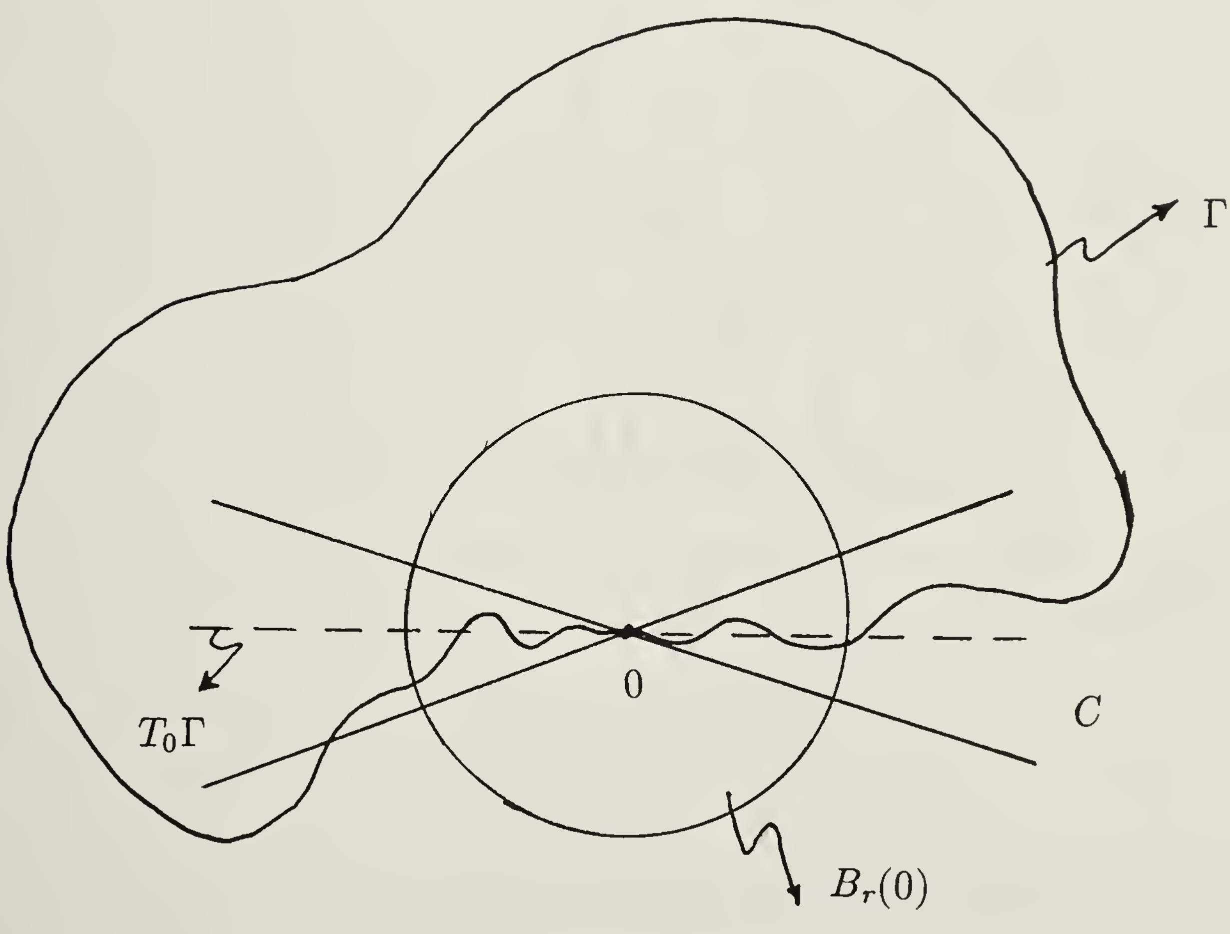
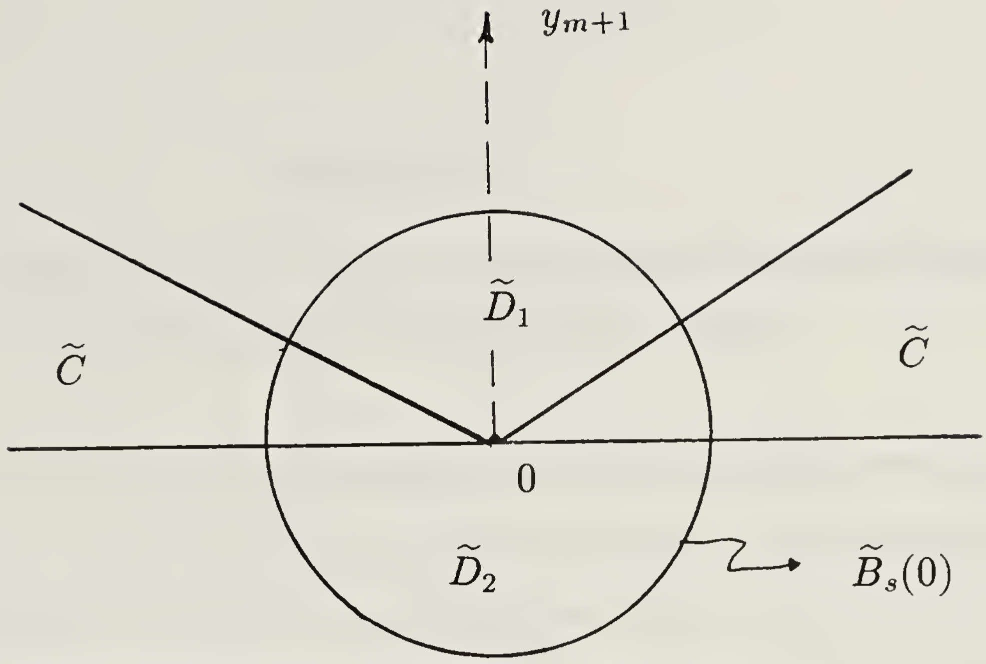
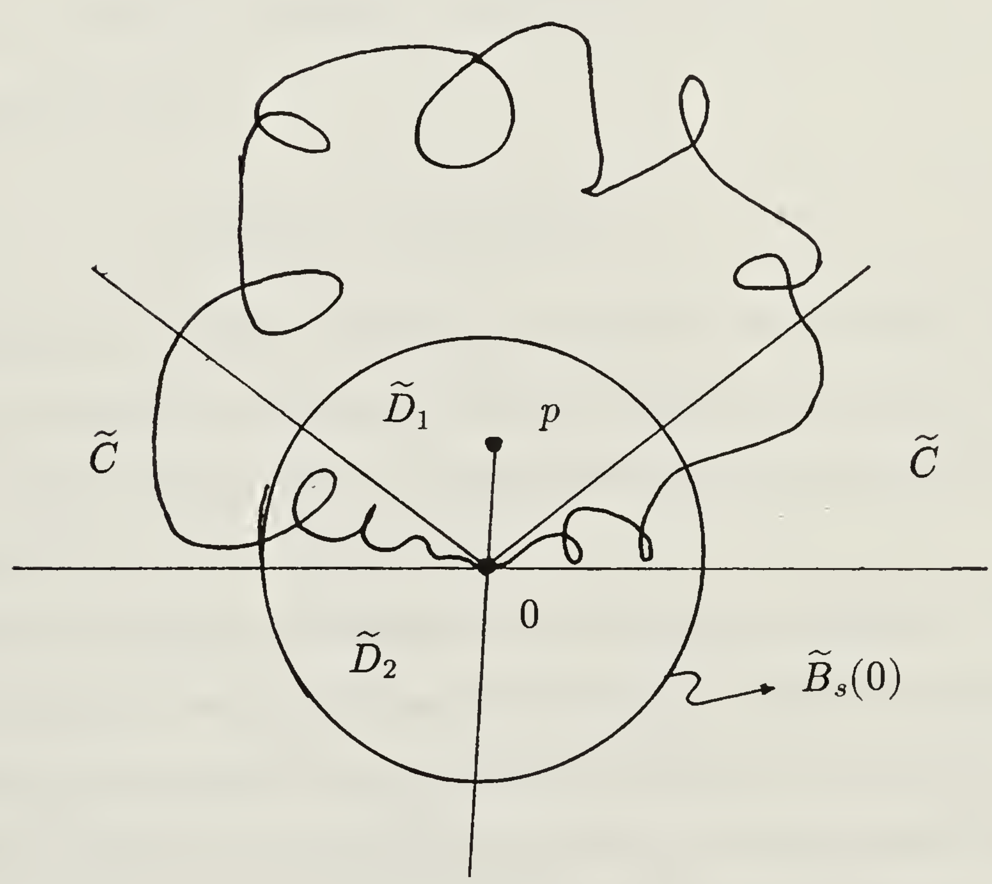


Figure 2.1.4. A cone C at 0.



(a) The boundary of $\widetilde{C}$ splits $\widetilde{B}_s(0)$.



(b) A ray emitting from P and passing through 0.

Figure 2.1.5.

CHAPTER III

UPPER ESTIMATE OF THE HAUSDORFF DIMENSION OF COMPACT INVARIANT SETS

In this chapter we discuss upper estimation for the Hausdorff dimension $\dim_H \mathbf{A}$ of the global attractor $\mathbf{A}$ of a dissipative dynamical system. The foundation for development on this subject was laid by a beautiful result of A. Douady and J. Oesterlé [3] for discrete systems. They proved that, for a C^1 mapping $F : \mathbf{R}^n \to \mathbf{R}^n$ and a compact set $K \subset \mathbf{R}^n$ such that $F(K) \supset K$, $\dim_H K \leq d$ provided $\omega_{d,K}(F) < 1$ where

$$\omega_{d,K}(F) = \sup_{x \in K} \omega_{d,x}(F)$$

and

$$\omega_{d,x}(F) = \alpha_1(x) \cdot \cdots \cdot \alpha_k(x)\, \alpha_{k+1}^s(x)$$

with $k = [d]$, $s = d - k$, and

$$\alpha_1(x) \geq \cdots \geq \alpha_n(x) \geq 0$$

are the singular values of $DF(x)$, the tangent map of F at $x \in K$. This work was subsequently generalized to mappings in infinite dimensional spaces by P. Constantin, C. Foias and R. Temam [2], J. M. Ghidaglia and R. Temam [6], A. Eden, C. Foias, R. Temam [5] and P. Thieullen[10].

The idea used by Douady and Oesterlé to obtain upper estimates of $\dim_H K$ is to examine the change of volume in various dimensions under the tangent map $DF(x)$. Their result may be intuitively interpreted as follows: for an integer $k \geq 0$, if the k-dimensional volume is contracted by $DF(x)$ uniformly with respect to $x \in K$, then $\dim_H K \leq k$. The fractional upper bound is obtained by a more delicate argument involving the elliptic Hausdorff measure. In fact, $\omega_{k,x}(F)$ can be regarded as the l_2-operator norm of $\bigwedge^k DF(x)$, the k-th exterior power of $DF(x)$, which, from Appendix B, describes the change of k-dimensional volume

under the linear mapping $DF(x)$. One of our contributions to this subject is stated in Theorem 3.1.2. It allows the use of general vector norms of $\bigwedge^k DF(x)$, based on the fact that all such norms in a finite dimensional space are equivalent. As we will see in Section 3.2, this leads to more easily computable and often better estimates for $\dim_H K$, since singular values are often difficult to compute.

As we have seen in Chapter I, periodic differential equations in $\mathbf{R}^n$ generate a discrete dynamical system in $\mathbf{R}^n$. The result of Douady and Oesterlé can thus be applied to obtain upper estimates for the Hausdorff dimension of compact invariant sets (including limit sets and the global attractor) of dissipative periodic ordinary differential equations. This was first done by R. A. Smith [8]. He also considered general nonautonomous equations. Our method presented here is to use the compound equations to study the evolution of volumes under the flow of differential equations. This method as well as a general condition of Dulac type derived from it given in (3.11) and its weaker form stated in Theorem 3.2.4 will play a very important role throughout this thesis. The theory of compound matrices and compound equations is outlined in Appendix B.

In Section 3.1, we state the result of Douady and Oesterlé for discrete systems; in Section 3.2, applications to ordinary differential equations are discussed; in Section 3.3, we present a more detailed study for autonomous differential equations, where more general techniques are developed and more general conditions derived. These techniques and conditions will be used in Chapter IV to develop higher dimensional Dulac criteria. Also in this section, possibilities of estimating the Hausdorff dimension of limit sets using information on the limiting trajectories are discussed. Finally in Section 3.4, as an illustration of our method, upper estimates for the Lorenz attractor are obtained. Our estimates are comparable to those of A. Eden [4] and V. A. Boichenko and G. A. Leonov[1].

§3.1. A result of Douady and Oesterlé for Discrete Dynamical Systems

Let U be an open subset of $\mathbf{R}^n$ and $F : U \to \mathbf{R}^n$ be a C^1 mapping. For each

$x \in U$, let $DF(x)$ denote the tangent map of F at x. $DF(x)$ is a bounded linear operator on $\mathbf{R}^n$ for each fixed x and depends continuously on x in the topology given by the induced operator norm. Let

$$\alpha_1(x) \geq \alpha_2(x) \geq \cdots \geq \alpha_n(x) \geq 0$$

be the singular values of $DF(x)$, i.e., eigenvalues of the square root of the positive semidefinite operator $DF(x)^* DF(x)$.

Suppose that $K \subset U$ is compact, and d is a real number, $0 \leq d \leq n$. Write $d = k + s$ with $k = [d]$ and $s = d - k$, the integer and noninteger part of d, respectively. Define

$$\omega_{d,x}(F) = \alpha_1(x) \cdots \alpha_k(x) \alpha_{k+1}^s(x)$$

and

$$\omega_{d,K}(F) = \sup_{x \in K} \omega_{d,x}(F).$$

The following result, due to A. Douady and J. Oesterlé [3], provides an upper estimate of the Hausdorff dimension of a negatively invariant compact set. Its proof can be found in [3] or [9].

Theorem 3.1.1. (*Douady - Oesterlé*) *Suppose K is a compact set such that $F(K) \supset K$. If $\omega_{d,K}(F) < 1$ for some $0 \leq d \leq n$, then $\dim_H K \leq d$. Moreover, if $d < 1$ then $\dim_H K = 0$.*

Remarks.

(i). Since F is C^1, the condition $\omega_{d,K}(F) < 1$ actually implies the strict inequality $\dim_H K < d$. In particular, the d-dimensional Hausdorff measure of K is zero.

(ii). From the Appendix B we know that when d is an integer $\omega_{d,x}(F)$ is the matrix norm of $\bigwedge^k DF(x)$, the k-th exterior power of the linear operator $DF(x)$ induced from the l_2 norm in $\mathbf{R}^n$. For $d = k + s$, the following is obvious

$$\omega_{d,x}(F) = \omega_{k,x}^{1-s}(F) \omega_{k+1,x}^s(F).$$

(iii). The assumption $\omega_{d,K}(F) < 1$ implies that

$$\omega_{d,x}(F^m) \to 0 \qquad \text{as} \quad m \to \infty$$

exponentially and the exponential rate is uniform with respect to $x \in K$. As a matter of fact, Theorem 3.1.1 can be proved under this weaker condition.

(iv). Theorem 3.1.1 has been generalized to dynamical systems in infinite dimensional spaces. For the case of Hilbert spaces, this was done by P. Constantin, C. Foias, and R. Temam [2] and by J. M. Ghidaglia and R. Temam [6]. Generalizations to general Banach spaces were given by a recent work of P. Thieullen [10].

Now let $|\cdot|_k$ and $|\cdot|_{k+1}$ denote general vector norms in $\bigwedge^k \mathbf{R}^n$ and $\bigwedge^{k+1} \mathbf{R}^n$, as well as the matrix norms they induce in the spaces of $\binom{n}{k} \times \binom{n}{k}$ and $\binom{n}{k+1} \times \binom{n}{k+1}$ matrices, respectively. Define

$$\Omega_{d,x}(F) = \Big|\bigwedge^k DF(x)\Big|_k^{1-s} \cdot \Big|\bigwedge^{k+1} DF(x)\Big|_{k+1}^{s} \qquad \text{for} \quad x \in K$$

and

$$\Omega_{d,K}(F) = \sup_{x \in K} \Omega_{d,x}(F).$$

Obviously, when both $|\cdot|_k$ and $|\cdot|_{k+1}$ are l_2 norms, $\Omega_{d,K}(F)$ agrees with $\omega_{d,K}(F)$.

Suppose $\Omega_{d,K}(F) < 1$. Then

$$\Omega_{d,x}(F^m) \to 0 \qquad \text{as} \quad m \to \infty$$

exponentially and the exponential rate is uniform with respect to $x \in K$. Since all vector norms on a finite dimensional vector space are equivalent, this would imply

$$\omega_{d,x}(F^m) \to 0 \qquad \text{as} \quad m \to \infty$$

exponentially with a uniform exponential rate with respect to $x \in K$. Using the remark (iii) following Theorem 3.1.1 we arrive at the following generalization of Theorem 3.1.1.

Theorem 3.1.2. *Suppose that* $K \subset U$ *is a compact set such that* $F(K) \supset K$. *If* $\Omega_{d,K}(F) < 1$ *for some* $0 \le d \le n$, *then* $\dim_H K < d$. *Moreover, if* $d < 1$ *then* $\dim_H K = 0$.

If the mapping F is dissipative with a bounded absorbing set $B \subset U$, then $\mathbf{A} = \omega(B)$ is the global attractor in U. In particular, $\mathbf{A}$ is compact and invariant under F. The following upper estimate for $\dim_H \mathbf{A}$ follows from Theorem 3.1.2.

Theorem 3.1.3. *If* $\Omega_{d,\mathbf{A}}(F) < 1$, *then* $\dim_H \mathbf{A} < d$. *Moreover, if* $d < 1$ *then* $\dim_H \mathbf{A} = 0$.

Remark. From Corollary 2.2.9 of Chapter II, we also know that when $\dim_H \mathbf{A} < 1$, $\mathbf{A} = \{\bar{x}\}$, where $\bar{x}$ is a globally asymptotically stable fixed point of F.

§3.2. Compact Invariant Sets of Ordinary Differential Equations

Let $D \subset \mathbf{R}^n$ be an open set and the mapping $(t, x) \mapsto f(t, x) \in \mathbf{R}^n$ be C^1 in $\mathbf{R} \times D$. We consider the general nonautonomous system of ordinary differential equations

$$x' = f(t, x). \tag{2.1}$$

Let $x(t, x_0)$ denote the unique solution to (2.1) such that $x(0, x_0) = x_0$, and $\frac{\partial f}{\partial x}$ be the Jacobian matrix of f. The linear variational equation of (2.1) with respect to $x(t, x_0)$ is given by

$$y'(t) = \frac{\partial f}{\partial x}\big(t,\, x(t, x_0)\big)\, y(t). \tag{2.2}$$

Suppose that there exists an open set $D_0 \subset D$ such that $x(t, x_0)$ exists and $x(t, x_0) \in D_0$ for all $0 \le t \le \tau$ for some $\tau > 0$ when $x_0 \in D_0$. Then we can define, for all $0 \le t \le \tau$, a mapping $F_t : D_0 \to D$ by $F_t(x_0) = x(t, x_0)$. Its tangent map $DF_t(x_0) = \frac{\partial x}{\partial x_0}(t, x_0)$ satisfies the linear system (2.2). From the Appendix B we know that the k-th exterior power $z(t) = \bigwedge^k DF_t(x_0)$ of the

linear mapping $DF_t(x_0)$ satisfies the k-th compound equation of (2.2)

$$z'(t) = \frac{\partial f}{\partial x}^{[k]} \big(t,\, x(t,x_0)\big)\, z(t). \tag{2.3}$$

We denote by $|\cdot|_k$ and $|\cdot|_{k+1}$ the vector norm in $\bigwedge^k \mathbf{R}^n$ and $\bigwedge^{k+1} \mathbf{R}^n$, as well as the induced matrix norm on the space of $\binom{n}{k} \times \binom{n}{k}$ and $\binom{n}{k+1} \times \binom{n}{k+1}$ matrices, respectively. We also let μ_k and μ_{k+1} denote the Lozinskiĭ measures corresponding to $|\cdot|_k$ and $|\cdot|_{k+1}$, respectively. From Theorem A.3.1 of the Appendix A we have the following inequality:

$$\Big|\overset{i}{\bigwedge} DF_t(x_0)\Big|_i \leq \exp \int_0^t \mu_i \left(\frac{\partial f}{\partial x}^{[i]} (\lambda, x(\lambda, x_0))\right) d\lambda \tag{2.4}$$

for all $x_0 \in D_0$ and $0 \leq t \leq \tau$, $i = k,\, k+1$. For simplicity of notation, we write (2.4) as

$$\Big|\overset{i}{\bigwedge} DF_t(x_0)\Big|_i \leq \exp \int_0^t \mu_i \left(\frac{\partial f}{\partial x}^{[i]}\right).$$

Thus, in this convention (2.4) leads to

$$\Big|\overset{k}{\bigwedge} DF_t(x_0)\Big|_k^{1+s} \Big|\overset{k+1}{\bigwedge} DF_t(x_0)\Big|_{k+1}^{s} \leq$$

$$\exp \int_0^t \left[(1-s)\,\mu_k\Big(\frac{\partial f}{\partial x}^{[k]}\Big) + s\,\mu_{k+1}\Big(\frac{\partial f}{\partial x}^{[k+1]}\Big) \right] \tag{2.5}$$

for all $x_0 \in D_0$ and $0 \leq t \leq \tau$. Theorem 3.1.2 and (2.5) yield the following result which provides an upper estimate for the Hausdorff dimension of compact negatively invariant sets of F_τ.

Theorem 3.2.1. *Suppose that there exists an integer $k \in [0, n-1]$ and a real number $s \in [0,1]$ such that*

$$\int_0^\tau \left[(1-s)\,\mu_k \left(\frac{\partial f}{\partial x}^{[k]}\right) + s\,\mu_{k+1} \left(\frac{\partial f}{\partial x}^{[k+1]}\right) \right] < 0 \tag{2.6}$$

for all x_0 in a compact set $K \subset D$. If $F_\tau(K) \supset K$, then $\dim_H K < k + s$.

Remark. The norms $|\cdot|_k$ and $|\cdot|_{k+1}$ in Theorem 3.2.1 are not necessarily the same. If both $|\cdot|_k$ and $|\cdot|_{k+1}$ are the l_2 norms, then Theorem 3.2.1 is first proved by R. A. Smith [8].

Now suppose that the system (2.1) is ω-periodic, i.e., for some $\omega > 0$

$$f(t+\omega, x) = f(t, x)$$

for all $(t,x) \in \mathbf{R} \times D$. Let $\mathcal{P}$ be the associated Poincaré map defined by $\mathcal{P}x_0 = x(\omega, x_0)$. As a special case of Theorem 3.2.1, we have the following corollary.

Corollary 3.2.2. *Suppose that $K \subset D$ is a compact set such that $\mathcal{P}(K) \supset K$ and that there exists an integer $k \in [0, n-1]$ and a real number $s \in [0,1]$ such that*

$$\int_0^\omega \left[(1-s)\,\mu_k\left(\frac{\partial f}{\partial x}^{[k]}\right) + s\,\mu_{k+1}\left(\frac{\partial f}{\partial x}^{[k+1]}\right) \right] < 0 \tag{2.7}$$

for all $x_0 \in K$. Then $\dim_H K < k+s$.

In particular, suppose that $\mathcal{P}$ is dissipative and $\mathbf{A} \subset D$ is the global attractor. Then $\dim_H \mathbf{A} < k+s$ if (2.7) holds on $\mathbf{A}$.

Next we assume that (2.1) is autonomous, namely, $f(t,x)$ is a function of x only. Let ϕ_t be the flow generated by (2.1). Suppose $K \subset D$ is a compact set invariant under the autonomous system (2.1). Then $\phi_t(K) = K$ for all $t \in \mathbf{R}$. The following corollary is a direct result of applying Theorem 3.2.1 to ϕ_t.

Corollary 3.2.3. *Suppose that (2.1) is autonomous and that $K \subset D$ is a compact invariant set. If there exists an integer $k \in [0, n-1]$ and a real number $s \in [0,1]$ such that*

$$\int_0^t \left[(1-s)\,\mu_k\left(\frac{\partial f}{\partial x}^{[k]}\right) + s\,\mu_{k+1}\left(\frac{\partial f}{\partial x}^{[k+1]}\right) \right] < 0 \tag{2.8}$$

for all $x_0 \in K$ and for some $t > 0$, then $\dim_H K < k+s$.

In the case when (2.1) is dissipative with the global attractor $\mathbf{A} \subset D$, $\mathbf{A}$ is

the maximal compact invariant set. Therefore $\dim_H \mathbf{A} < k + s$ if (2.8) holds for all $x \in \mathbf{A}$.

For a compact invariant set K, define

$$q_k(K,\, t) \;=\; \sup_{x_0 \in K} \frac{1}{t} \int_0^t \mu_k \left(\frac{\partial f^{[k]}}{\partial x} (x(\lambda,\, x_0)) \right) d\lambda \tag{2.9}$$

and

$$q_k(K) \;=\; \limsup_{t \to \infty} q_k(K,\, t). \tag{2.10}$$

Since K is compact and invariant, these quantities are well-defined. We would like to point out that q_k obviously depends on the choice of the vector norm $|\cdot|_k$ in $\mathbf{R}^N$, $N = \binom{n}{k}$, and the corresponding Lozinskiĭ measure μ_k. When the norm $|\cdot|_k$ is chosen as the l_2 norm, the quantity q_k has also been considered by R. Temam [9] as an upper bound for the global Lyapunov exponents in the context of evolution equations in Hilbert spaces. Temam uses a trace formula involving projections to finite dimensional subspaces rather than the compound matrices. In the case of finite dimensional spaces, by choosing different vector norms in $\mathbf{R}^N$, we may obtain from (2.9) and (2.10) different and often easier calculations of q_k.

More generally, for $0 \le d \le n$, let $k = [d]$ and $s = d - k$ be the integer and noninteger part of d , respectively. We can define similarly the following quantities:

$$q_d(K,\, t) \;=\; \sup_{x_0 \in K} \frac{1}{t} \int_0^t \left[(1 + s)\, \mu_k \left(\frac{\partial f^{[k]}}{\partial x} \right) + s\, \mu_{k+1} \left(\frac{\partial f^{[k+1]}}{\partial x} \right) \right]$$

and

$$q_d(K) \;=\; \limsup_{t \to \infty} q_d(K,\, t).$$

Theorem 3.2.4. *Suppose that* (2.1) *is autonomous and that* K *is a compact invariant set. If* $q_d(K) < 0$ *for some* $0 \le d \le n$, *then* $\dim_H K < d$.

Proof. Suppose $q_d(K) < 0$. Then, for some $\delta > 0$, there exists a $t > 0$ such that

$$\sup_{x_0 \in K} \frac{1}{t} \int_0^t \left[(1 - s)\, \mu_k (\frac{\partial f^{[k]}}{\partial x}) + s\, \mu_{k+1} (\frac{\partial f^{[k+1]}}{\partial x}) \right] \le -\delta < 0,$$

which implies that (2.8) holds for all $x_0 \in K$. Therefore from Corollary 3.2.3 we know that $\dim_H K < d$. □

Corollary 3.2.5. *Suppose* $\mathbf{A}$ *is the global attractor of the autonomous system* (2.1). *If* $q_d(\mathbf{A}) < 0$ *for some* $0 \le d \le n$, *then* $\dim_H \mathbf{A} < d$.

§3.3. Autonomous Systems

In this section we give more detailed treatment on the upper estimate of the Hausdorff dimension of compact invariant sets of autonomous systems. Applications of some of the techniques and results in this section will be seen in Chapter IV where we derive Bendixson's criterion for dissipative autonomous systems in $\mathbf{R}^n$.

Let $D \subset \mathbf{R}^n$ be an open set and $x \mapsto f(x) \in \mathbf{R}^n$ be a C^1 mapping defined for all $x \in D$. Consider the autonomous system

$$x' = f(x). \tag{3.1}$$

Let $x(t, x_0)$ denote the unique solution to (3.1) such that $x(0, x_0) = x_0$. Then the k-th compound equation of the linear variational equation of (3.1) with respect to $x(t, x_0)$ is given by

$$y'(t) = \frac{\partial f^{[k]}}{\partial x}\big(x(t, x_0)\big)\, y(t). \tag{3.2}$$

where $\frac{\partial f}{\partial x}$ is the Jacobian matrix of f, and $\frac{\partial f}{\partial x}^{[k]}$ is the k-th additive compound matrix of $\frac{\partial f}{\partial x}$ (see Appendix B).

Consider a real-valued function $(x, y) \mapsto V(x, y)$ defined for $(x, y) \in D \times \mathbf{R}^N$, $N = \binom{n}{k}$. We assume that V is locally Lipschitz continuous in its domain and

$$\lim_{h \to 0+} \frac{1}{h}\left[V(x + ha,\, y + hb) - V(x,\, y)\right]$$

exists for all $(x, y) \in D \times \mathbf{R}^N$ and all $(a, b) \in \mathbf{R}^n \times \mathbf{R}^N$.

For each $(x, y) \in D \times \mathbf{R}^N$, we define $\dot{V}(x, y)$ by

$$\dot{V}(x, y) = \lim_{h \to 0+} \frac{1}{h}\left[V\left(x + hf(x),\; y + h\frac{\partial f^{[k]}}{\partial x}(x)y\right) - V(x,\, y)\right]. \tag{3.3}$$

Then

$$\overset{\bullet}{V}(x,y) = \frac{\partial V^*}{\partial x} f(x) + \frac{\partial V^*}{\partial y} \frac{\partial f^{[k]}}{\partial x}(x)\,y \tag{3.4}$$

almost everywhere since V is lipschitzian and therefore differentiable almost everywhere. When $k = n$,

$$\overset{\bullet}{V}(x,y) = \frac{\partial V^*}{\partial x} f(x) + \frac{\partial V}{\partial y}(\mathrm{div} f)\,y. \tag{3.5}$$

Suppose now that $x(t)$ is a solution to (3.1) and $y(t)$ is a solution of the k-th compound equation (3.2). Let $V(t) = V(x(t),\, y(t))$. We have the following result.

Lemma 3.3.1. $D_t^+ V(t) = \overset{\bullet}{V}\big(x(t),\, y(t)\big)$ *for almost all* $t > 0$.

Proof. Since $x(t)$ and $y(t)$ satisfy (3.1) and (3.2), respectively, for sufficiently small $h > 0$,

$$x(t+h) = x(t) + h\, f(x(t)) + o(h)$$

$$y(t+h) = y(t) + h\,\frac{\partial f^{[k]}}{\partial x}(x(t))\, y(t) + o(h).$$

Thus

$$\begin{aligned} D_t^+ V(t) &= \lim_{h\to 0+} \frac{1}{h}\left[V\big(x(t+h),\, y(t+h)\big) - V\big(x(t),\, y(t)\big)\right] \\ &= \lim_{h\to 0+} \frac{1}{h}\Bigg[V\Big(x(t) + h\, f(x(t)),\, y(t) + h\,\frac{\partial f^{[k]}}{\partial x}(x(t))\, y(t)\Big) \\ &\qquad - V\big(x(t),\, y(t)\big) + R(t,\, h)\Bigg] \end{aligned}$$

where

$$\begin{aligned} R(t,\, h) = {}& V\Big(x(t) + h\, f(x(t)) + o(h),\; y(t) + h\,\frac{\partial f^{[k]}}{\partial x}(x(t))\, y(t) + o(h)\Big) - \\ & V\Big(x(t) + h\, f\big(x(t)\big),\; y(t) + h\,\frac{\partial f^{[k]}}{\partial x}\big(x(t)\big)\, y(t)\Big). \end{aligned}$$

It follows from the Lipschitzian continuity of V that

$$\lim_{h\to 0+} \frac{1}{h} R(t,\, h) = 0$$

for almost all $t > 0$. As the result,

$$D_t^+ V(t) = \overset{\bullet}{V}\big(x(t),\, y(t)\big).$$

□

Theorem 3.3.2. *Suppose that $K \subset D$ is a compact invariant set and that there exist constants $a, b > 0$ and a function $V(x, y)$ such that*

(1) $V(x,\, y) \geq a\,|y|_k$,

(2) $\overset{\bullet}{V}(x,\, y) \leq -b\,|y|_k$,

for all $x \in K$ and $y \in \mathbf{R}^N$, $N = \binom{n}{k}$. Then $\dim_H K < k$.

Proof. For each $x_0 \in K$, consider the solution $x(t, x_0)$ of (3.1) and a solution $y(t)$ of the k-th compound equation (3.2). Let $V(t) = V(x(t), y(t))$. Then Lemma 3.3.1 gives us

$$D_t^+ V(t) = \overset{\bullet}{V}\big(x(t),\, y(t)\big) \leq -b\,|y(t)|_k \quad \text{for all } t > 0.$$

Thus

$$a\,|y(t)|_k \leq V(t) \leq V(0) - b \int_0^t |y(s)|_k\, ds.$$

Therefore

$$|y(t)|_k \leq \frac{1}{a}\, V(0) - \frac{b}{a} \int_0^t |y(s)|_k\, ds.$$

Gronwall's Lemma implies

$$|y(t)|_k \leq \frac{1}{a}\, V(0) \exp\big(-\frac{bt}{a}\big). \tag{3.6}$$

Using a similar argument to that used in the proof of Theorem 3.1.2 we can show that (3.6) implies $\dim_H K < k$. □

For practical purposes, we are interested in a general class of V given by

$$V(x, y) = |A(x)\,y| \tag{3.7}$$

where $|\cdot|$ is a vector norm in $\mathbf{R}^N$, $N = \binom{n}{k}$, and $x \mapsto A(x)$ is a $N \times N$ matrix-valued function. We assume that A is C^1 and $A(x)$ is nonsingular for all $x \in K$. As a result, there exists a constant $c > 0$ such that

$$V(x, y) \geq c\,|y| \tag{3.8}$$

for all $x \in K$ and $y \in \mathbf{R}^N$.

Now it follows from the definition of $\overset{\bullet}{V}$ that

$$-\mu(-B)V \leq \overset{\bullet}{V} \leq \mu(B)V \tag{3.9}$$

where

$$B = A_f\,A^{-1} + A\frac{\partial f}{\partial x}^{[k]} A^{-1} \tag{3.10}$$

and μ is the Lozinskiĭ measure corresponding to the $|\cdot|$ in $\mathbf{R}^N$, and A_f is the matrix obtained by replacing each entry a_{ij} of A by $\frac{\partial a_{ij}}{\partial x}^* f$, its directional derivative in the direction of f.

When $A = I$, we have $B = \frac{\partial f}{\partial x}^{[k]}$ and $\mu(B) = \lambda_1 + \cdots + \lambda_k$ and $-\mu(-B) = \lambda_{n-k+1} + \cdots + \lambda_n$ in the case when $|y| = |y^* y|^{\frac{1}{2}}$, where $\lambda_1 \leq \cdots \leq \lambda_n$ are eigenvalues of the symmetric matrix $(\frac{\partial f}{\partial x}^* + \frac{\partial f}{\partial x})/2$. When $k = n$, $-\mu(-B) = \mu(B) = \operatorname{div} f$, so that $\overset{\bullet}{V} = \operatorname{div} f\, V$, which is the familiar formula of Liouville and Jacobi (see [7]).

Theorem 3.3.3. *Suppose $K \subset D$ is a compact invariant set. If*

$$\mu\big(A_f\,A^{-1} + A\frac{\partial f}{\partial x}^{[k]} A^{-1}\big) < 0 \tag{3.11}$$

on K, then $\dim_H K < k$.

Proof. Since K is compact, there exists a $\delta > 0$ such that (3.11) implies

$$\mu\big(A_f\,A^{-1} + A\frac{\partial f}{\partial x}^{[k]} A^{-1}\big) \leq -\delta < 0 \quad \text{on } K.$$

Then (3.9) implies

$$\dot{V}(x,y) \le -\delta V(x,y) \le -b\,|y|$$

for all $x \in K$ and $y \in \mathbf{R}^N$, where

$$b = \delta \max_{x \in K} |A(x)|.$$

Thus $V(x,y)$ satisfies the assumptions (1) and (2) in Theorem 3.3.2. Therefore $\dim_H K < k$. □

Remark. Theorem 3.3.3 still holds if the condition (3.11) is replaced by

$$\mu\big(-A_f\, A^{-1} - A\frac{\partial f^{[k]}}{\partial x} A^{-1}\big) < 0 \quad \text{on } K.$$

This can be shown using a time reversal argument.

When $A = I$, Theorem 3.3.3 yields the following corollary.

Corollary 3.3.4. *Suppose that one of*

$$\mu\big(\frac{\partial f^{[k]}}{\partial x}\big) < 0, \qquad \mu\big(-\frac{\partial f^{[k]}}{\partial x}\big) < 0 \tag{3.12}$$

holds on a compact invariant set K. *Then* $\dim_H K < k$.

Corresponding to the l_∞, l_1, and l_2 norm of $\mathbf{R}^N$, $N = \binom{n}{k}$, the conditions in

(3.12) take the following form:

(i) $$\sum_{r\in(i)} \frac{\partial f_r}{\partial x_r} + \sum_{\substack{q\notin(i)\\ r\in(i)}} \left|\frac{\partial f_q}{\partial x_r}\right| < 0$$

for all k-tuples $(i) = (i_1, \cdots, i_k)$, $1 \le i_1 < \cdots < i_k \le n$,

(ii) $$\sum_{r\in(i)} \frac{\partial f_r}{\partial x_r} + \sum_{\substack{q\notin(i)\\ r\in(i)}} \left|\frac{\partial f_r}{\partial x_q}\right| < 0$$

for all k-tuples $(i) = (i_1, \cdots, i_k)$, $1 \le i_1 < \cdots < i_k \le n$,

(iii) $\lambda_1 + \cdots + \lambda_k < 0,$

(iv) $$\sum_{r\in(i)} \frac{\partial f_r}{\partial x_r} - \sum_{\substack{q\notin(i)\\ r\in(i)}} \left|\frac{\partial f_q}{\partial x_r}\right| > 0$$

for all k-tuples $(i) = (i_1, \cdots, i_k)$, $1 \le i_1 < \cdots < i_k \le n$,

(v) $$\sum_{r\in(i)} \frac{\partial f_r}{\partial x_r} - \sum_{\substack{q\notin(i)\\ r\in(i)}} \left|\frac{\partial f_r}{\partial x_q}\right| > 0$$

for all k-tuples $(i) = (i_1, \cdots, i_k)$, $1 \le i_1 < \cdots < i_k \le n$.

(vi) $\lambda_{n-k+1} + \cdots + \lambda_n > 0,$

where $\lambda_1 \ge \cdots \lambda_n$ are eigenvalues of the symmetric matrix $(\frac{\partial f}{\partial x}^* + \frac{\partial f}{\partial x})/2$. (iii) and (vi) give Smith's conditions, which may be difficult to compute. Conditions (i) (ii) (iv) and (v), however, can be easily computed from the equation (3.1). When $A(x) \neq I$, the corresponding expressions are more tedious but not more difficult to calculate and they provide the added flexibility of having N^2 arbitrary functions (entries of $A(x)$) at our disposal. This flexibility is exploited in Chapter VII where we apply these techniques to biological models.

When $k = n$, $A(x)$ is a real-valued function. We thus have the following corollary of Theorem 3.3.3.

Corollary 3.3.5. *Suppose there exists a real-valued function* $\alpha > 0$ *such that*

$$\frac{\partial \alpha}{\partial x}^{*} f + \alpha \, \mathrm{div} f < 0. \tag{3.13}$$

Then $\dim_H K < n$.

Remarks.

(i). Condition (3.13) can be rewritten as

$$\mathrm{div}\big(\alpha f\big) < 0 \quad \text{on } K, \tag{3.14}$$

which is a condition of Dulac type.

(ii). Corollary 3.3.5 still holds if (3.13) is replaced by

$$\mathrm{div}\big(\alpha f\big) > 0 \quad \text{on } K.$$

Sometimes a weaker form of the condition (3.11) is desirable. Assume that K is a compact invariant set for (3.1), and let k be an integer and $A(x)$ be the $N \times N$ matrix-valued function considered in (3.7). Define the following quantities which are generalizations of $q_k(K,t)$ and $q_k(K)$ defined in (2.10):

$$\bar{q}_k(K,t) = \sup_{x_0 \in K} \frac{1}{t} \int_0^t \mu\left(B\big(x(s,x_0)\big) \right) ds \tag{3.15}$$

with the $N \times N$ matrix B given in (3.10), and

$$\bar{q}_k(K) = \limsup_{t \to \infty} \bar{q}_k(K,t). \tag{3.16}$$

Obviously when $A = I$, $\bar{q}_k(K) = q_k(K)$.

Now suppose $\bar{q}_k(K) = -\delta < 0$ for some function $A(x)$ and integer $0 \le k \le n$. Then (3.15) and (3.16) lead to

$$\int_0^t \mu\left(\big(x(s,x_0)\big) \right) ds \le -\delta t \quad \text{for all } x_0 \in K. \tag{3.17}$$

Let $V(x,y)$ be the function defined in (3.7). Let $x(t) = x(t, x_0)$ and $y(t)$ be a solution to the k-th compound equations (3.2). Then, from (3.8), there exists a constant $c > 0$ such that

$$V(t) =: V\big(x(t), y(t)\big) \geq c\,|y(t)| \qquad \text{for all} \quad t > 0.$$

Lemma 3.3.1 and (3.9) imply that

$$D_t^+ V(t) = \dot{V}\big(x(t), y(t)\big) \leq \mu(B)\,|y(t)|$$

for almost all $t > 0$. Therefore

$$c\,|y(t)| \leq V(t) \leq V(0) + \int_0^t \mu(B)\,|y(s)|\,ds$$

for all $t > 0$. By Gronwall's lemma we have

$$|y(t)| \leq \frac{1}{c}\,V(0)\,\exp \int_0^t \mu(B)\,ds \leq \frac{1}{c}\,V(0)\,\exp(-\delta\,t)$$

for all $t > 0$, which implies that $|y(t)| \to 0$ as $t \to \infty$ exponentially with a uniform exponential rate with respect to $x_0 \in K$. The same argument as in the proof of Theorem 3.3.2 leads to the following result.

Theorem 3.3.6. *Suppose K is a compact invariant set for (3.1) . Then $\dim_H K < k$ provided $\bar{q}_k(K) < 0$ for some integer $0 \leq k \leq n$ and $N \times N$ matrix-valued function $A(x)$.*

In the following, the Hausdorff dimension of limit sets to (3.1) will be estimated from above using information along the limiting trajectories.

Let $B \subset D$ be a nonempty subset. We assume that $\gamma^+(B)$ is bounded. Then from Lemma 1.2.1 in Chapter I we know that $\omega(B)$ is nonempty, compact, invariant, and attracts B.

Let ϕ_t denote the flow generated from (3.1). For a $T > 0$, we define

$$p_k(t, T, B) = \sup_{x \in B} \int_t^{t+T} \mu\left(\frac{\partial f^{[k]}}{\partial x}\,(\phi_\tau(x))\right)\,d\tau \tag{3.18}$$

and

$$p_k(T,B) = \limsup_{t\to\infty} p_k(t,T,B). \tag{3.19}$$

These quantities are well-defined from our assumptions on B.

Theorem 3.3.7. *Suppose $\gamma^+(B)$ is bounded. If $p_k(T,B) < 0$ for some $T > 0$ and integer k, then $\dim_H \omega(B) < k$.*

Proof. Suppose $\bar{x} \in \omega(B)$. Then there is a sequence of points x_n in B and a sequence of real numbers $t_n > 0$ such that $\phi_{t_n}(x_n) \to \bar{x}$, as $n \to \infty$. Let $y_n = \phi_{t_n}(x_n)$, then for any $t > 0$, $\phi_t(y_n) \to \phi_t(\bar{x})$ and $D\phi_t(y_n) \to D\phi_t(\bar{x})$ as $n \to \infty$, which implies $D\phi_t(y_n)^{(k)} \to D\phi_t(\bar{x})^{(k)}$ as $n \to \infty$. Now $\phi_{t+t_n}(x_n) = \phi_t(\phi_{t_n}(x_n))$. Thus by the chain rule

$$D\phi_{t+t_n}(x_n) = D\phi_t(y_n)\, D\phi_{t_n}(x_n).$$

Therefore

$$D\phi_t(y_n) = D\phi_{t+t_n}(x_n)\, D\phi_{t_n}(x_n)^{-1}.$$

which implies the following relation for the k-th compound matrices,

$$D\phi_t(y_n)^{(k)} = D\phi_{t+t_n}(x_n)^{(k)} \left(D\phi_{t_n}(x_n)^{-1}\right)^{(k)}.$$

Let $Y(t) = D\phi_{t+t_n}(x_n)\, D\phi_{t_n}(x_n)^{-1}$. Then for Appendix B,

$$\frac{d}{dt}Y(t) = \frac{\partial f}{\partial x}\left(\phi_{t+t_n}(x_n)\right) Y(t)$$

and

$$\frac{d}{dt}Y^{(k)}(t) = \frac{\partial f}{\partial x}^{[k]}\left(\phi_{t+t_n}(x_n)\right) Y^{(k)}(t).$$

For a vector norm $|\cdot|$ in $\mathbf{R}^N$, $N = \binom{n}{k}$, and the corresponding Lozinskiĭ measure μ, we have

$$\begin{aligned}\left|D\phi_T(y_n)^{(k)}\right| &\le \exp \int_0^T \mu\left(\frac{\partial f}{\partial x}^{[k]}\left(\phi_{\tau+t_n}(x_n)\right)\right) d\tau \\ &= \exp \int_{t_n}^{t_n+T} \mu\left(\frac{\partial f}{\partial x}^{[k]}\left(\phi_\tau(x_n)\right)\right) d\tau.\end{aligned}$$

Thus

$$\left|D\phi_T(\bar{x})^{(k)}\right| = \lim_{n\to\infty}\left|D\phi_T(y_n)^{(k)}\right| \le e^{p_k(T,B)} < 1.$$

Therefore $\dim_H\omega(B) < k$ by Theorem 3.1.2. □

In particular, when $B = \{x_0\}$, we have the following corollary of Theorem 3.3.7.

Corollary 3.3.8. *Suppose $\gamma^+(x_0)$ is bounded. If, for some $T > 0$,*

$$\limsup_{t\to\infty}\int_t^{t+T} \mu\Big(\frac{\partial f}{\partial x}^{[k]}(x(\tau,x_0))\Big)\,d\tau < 0 \tag{3.20}$$

then $\dim_H\omega(x_0) < k$.

In the spirit of Theorem 3.3.7 and Theorem 3.1.2, we may define the following quantities:

$$p_d(t,T,B) =$$

$$\sup_{x\in B}\int_t^{t+T}\left[(1-s)\mu_k\left(\frac{\partial f}{\partial x}^{[k]}(\phi_\tau(x))\right) + s\mu_{k+1}\left(\frac{\partial f}{\partial x}^{[k+1]}(\phi_\tau(x))\right)\right]d\tau \tag{3.21}$$

and

$$p_d(T,B) = \limsup_{t\to\infty} p_d(t,T,B) \tag{3.22}$$

where $d = k+s$, and prove the following result using a similar argument to that in the proof of Theorem 3.3.7.

Theorem 3.3.9. *Suppose that $\gamma^+(B)$ is bounded. If $p_d(T,B) < 0$ for some $T > 0$ and $0 \le d \le n$, then $\dim_H\omega(B) < d$.*

Corollary 3.3.10. *Suppose the semi-orbit $\gamma^+(x_0)$ is bounded. If, for some $T > 0$,*

$$\limsup_{t\to\infty}\int_t^{t+T}\left[(1-s)\mu_k\left(\frac{\partial f}{\partial x}^{[k]}(\phi_\tau(x_0))\right) + s\mu_{k+1}\left(\frac{\partial f}{\partial x}^{[k+1]}(\phi_\tau(x_0))\right)\right]d\tau < 0 \tag{3.23}$$

then $\dim_H \omega(x_0) < k + s$.

§3.4. Dimension of the Lorenz Attractor

We consider the Lorenz equation

$$\begin{aligned} x' &= -\sigma x + \sigma y \\ y' &= r x - y - x z \\ z' &= -b z + x y \end{aligned} \tag{4.1}$$

where σ, r, b are three positive parameters. We have shown in Chapter I that (4.1) is dissipative and its global attractor $\mathbb{L}$ is contained in the region

$$D = \{ (x, y, z) \in \mathbf{R}^3 \ : \ |x| \le \rho, \ y^2 + (z - r)^2 \le \rho^2 \},$$

where $\rho = \frac{r\,b}{2\sqrt{b-1}}$.

Numerical evidence shows that for certain range of parameters, (4.1) demonstrates chaotic behaviour, and suggests that its global attractor $\mathbb{L}$ is a set with complex geometry. In this section we shall apply the theory developed in §3.2 and §3.3 to estimate the upper bound for the Hausdorff dimension of $\mathbb{L}$, which is one of the indicators of the complexity of the geometry of $\mathbb{L}$.

The Jacobian matrix J and its compounds are given in the following:

$$J = \begin{bmatrix} -\sigma & \sigma & 0 \\ r - z & -1 & -x \\ y & x & -b \end{bmatrix} \tag{4.2}$$

$$J^{[2]} = \begin{bmatrix} -\sigma - 1 & -x & 0 \\ x & -b - \sigma & \sigma \\ -y & r - z & -b - 1 \end{bmatrix} \tag{4.3}$$

$$J^{[3]} = \operatorname{tr} J = -\sigma - b - 1. \tag{4.4}$$

Since $J^{[3]}$ is a negative number, Corollary 3.3.4 implies that $\dim_H \mathbb{L} < 3$. On the other hand, numerical results suggest that (4.1) has many periodic solutions. As a result of Theorem 2.3.4 of Chapter II, we know $\dim_H \mathbb{L} \ge 2$.

Theorem 3.4.1. *Suppose* (4.1) *has periodic solutions. Then*

$$2 \leq \dim_H \text{Ł} < 3.$$

In what follows, we shall use the methods in Theorem 3.2.1 of §3.2 to obtain finer upper estimate of $\dim_H$Ł.

Let ϕ_t be the flow generated by (4.1). Then

$$\left|D\phi_t^{(2)}\right| \leq \exp \int_0^t \mu(J^{[2]})$$

$$\left|D\phi_t^{(3)}\right| \leq \exp\left(-(\sigma + b + 1)t\right).$$

Thus

$$\left|D\phi_t^{(2)}\right|^{1-s} \left|D\phi_t^{(3)}\right|^s \leq \exp \int_0^t \left[(1-s)\mu(J^{[2]}) - s(\sigma + b + 1)t\right].$$

Suppose now

$$\mu(J^{[2]}) \leq M \qquad \text{on} \quad \text{Ł}$$

and we choose s_0 so that

$$s_0 = \frac{M}{M + \sigma + b + 1}. \tag{4.5}$$

Then

$$\left|D\phi_t^{(2)}\right|^{1-s} \left|D\phi_t^{(3)}\right|^s < 1$$

for all $s < s_0$. Therefore $\dim_H$Ł $\leq 2 + s_0$.

To obtain M, we choose the norm in $\mathbf{R}^3 \cong \mathbf{R}^{\binom{3}{2}}$ as

$$|(u, v, w)| = \sqrt{u^2 + v^2} + \alpha |w|, \quad \alpha > 0. \tag{4.6}$$

Then from the Appendix A we know the Lozinskiĭ measure of $J^{[2]}$ corresponding to this norm can be estimated by

$$\begin{aligned}\mu(J^{[2]}) &\leq \max\left\{-(\sigma+1) + \frac{1}{\alpha}\sqrt{y^2 + (r-z)^2},\ -(b+1) + \alpha\sigma\right\} \\ &\leq \max\left\{-(\sigma+1) + \frac{1}{\alpha}\rho,\ -(b+1) + \alpha\sigma\right\}.\end{aligned}$$

Choose $\alpha > 0$ so that

$$-(\sigma+1) + \frac{1}{\alpha}\rho = -(b+1) + \alpha\,\sigma$$

i.e.

$$\sigma\,\alpha^2 + (\sigma - b)\,\alpha - \rho = 0. \tag{4.7}$$

Then

$$\mu(J^{[2]}) \leq -(b+1) + \alpha\,\sigma.$$

Now equation (4.7) yields

$$\begin{aligned}\alpha &= \frac{1}{2\sigma}\left[b - \sigma + \left((\sigma-b)^2 + 4\,\sigma\,\rho\right)^{\frac{1}{2}}\right]\\ &= \frac{1}{2\sigma}\left[b - \sigma + \left((\sigma-b)^2 + \frac{2\sigma r b}{\sqrt{b-1}}\right)^{\frac{1}{2}}\right]\end{aligned}$$

hence

$$M = \frac{1}{2}\left[-(\sigma+b+2) + \left((\sigma-b)^2 + \frac{2\sigma r b}{\sqrt{b-1}}\right)^{\frac{1}{2}}\right] \tag{4.8}$$

and (4.5) implies the following result.

Theorem 3.4.2.

$$\dim_H L \leq 2 + \frac{M}{m + \sigma + b + 1} \tag{4.9}$$

with M *given in (4.8).*

When $b = \frac{8}{3}$, $\sigma = 10$, $r = 28$, (4.9) yields

$$\dim_H \mathrm{L} \leq 2.4. \tag{4.10}$$

Different upper estimates than (4.9) may be obtained by choosing different vector norms $|(u, v, w)|$ than that in (4.6).

Upper estimates for $\dim_H \mathrm{L}$ are also obtained by R.A. Smith [8], R. Temam [9], A. Eden [4], and V. A. Boichenko and G. A. Leonov [1]. When the parameter values are $b = 8/3, \sigma = 10, r = 28$, the upper bound in (4.10) is better than

those of Smith and Temam, and is comparable to those of Eden and Boichenko and Leonov.

§3.5. Bibliography for Chapter III

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CHAPTER IV

CRITERIA OF BENDIXSON AND DULAC IN HIGHER DIMENSIONS

In this chapter we develop general methods for proving the nonexistence of certain types of structures which are invariant with respect to an autonomous system of differential equations in $\mathbf{R}^n$

$$x' = f(x) \tag{0.1}$$

where the function $x \mapsto f(x)$ is assumed to be C^1 for x in an open subset D of $\mathbf{R}^n$. These invariant structures include periodic orbits, homoclinic orbits, heteroclinic cycles, and higher dimensional invariant tori. These methods will produce concrete conditions which, compared with most existing results on this subject, are more general and flexible and easier to compute.

The work in this field dates back to a classic result of Bendixson [1] in 1901 for 2-dimensional systems, which states that, when $n = 2$, (0.1) has no nonconstant periodic solutions if $\operatorname{div} f \neq 0$ on $\mathbf{R}^2$. A result of Dulac [6] generalizes this to the statement that, if $n = 2$ and $\operatorname{div}(\alpha f) \neq 0$ on a simply connected open subset D of $\mathbf{R}^2$, where α is a continuous real-valued function on D, then there is no closed path of (1.1) which lies entirely in D. A flurry of results on the nonexistence of periodic solutions systems of higher dimensions appeared in the next several decades. Most of these results, however, deal with only special types of equations arising from mechanics and other applied fields. The first generalizations of the results of Bendixson and Dulac for general equations (0.1) of arbitrary dimension n seem to belong to R. A. Smith [9] [10] and J. S. Muldowney [8] in the late 1980's.

Smith shows in [9] that if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are the eigenvalues of $\frac{1}{2}(\frac{\partial f}{\partial x}^* + \frac{\partial f}{\partial x})$, where $\frac{\partial f}{\partial x}$ is the Jacobian matrix of f and the asterisk denotes transposition, then all bounded semi-orbits of (0.1) tend to an equilibrium if $\lambda_1 + \lambda_2 < 0$ on $\mathbf{R}^n$. In particular, no nonconstant periodic solution can exist under

these circumstances. Smith also treats situations where $\lambda_1 + \lambda_2 < 0$ holds only on subsets D of $\mathbf{R}^n$. More generally, Theorem 2 of [9] shows that the Hausdorff dimension of any compact invariant set of (0.1) is less than 2, if $\lambda_1 + \lambda_2 < 0$, and Theorem 5 then implies that no simple closed contour is invariant if (0.1) is dissipative. Thus, for example, a dissipative system satisfying Smith's condition $\lambda_1 + \lambda_2 < 0$ can have no homoclinic orbits.

In [8], Muldowney proves that if $\mu\left(\frac{\partial f}{\partial x}^{[2]}\right) < 0$ or $\mu\left(-\frac{\partial f}{\partial x}^{[2]}\right) < 0$, then (0.1) has no nonconstant periodic solutions. Here $\frac{\partial f}{\partial x}^{[2]}$ is the second additive compound discussed in the Appendix B, and μ is the Lozinskiĭ measure corresponding to a vector norm $|\cdot|$ as defined and discussed in Appendix A. When $|y| = (y^*y)^{\frac{1}{2}}$, $\mu\left(\frac{\partial f}{\partial x}^{[2]}\right) = \lambda_1 + \lambda_2$ so that $\mu\left(\frac{\partial f}{\partial x}^{[2]}\right) < 0$ is Smith's condition in this case; here $\mu\left(-\frac{\partial f}{\partial x}^{[2]}\right) < 0$ means $\lambda_{n-1} + \lambda_n > 0$. The advantage of using the Lozinskiĭ measure is in that other choices of norm often lead to expressions $\mu\left(\pm\frac{\partial f}{\partial x}^{[2]}\right)$ which are easier to calculate or estimate than eigenvalues. Results of both Smith and Muldowney reduce to that of Bendixson when $n = 2$.

The argument of [8] that the condition $\lambda_1 + \lambda_2 < 0$ implies the nonexistence of closed paths is roughly as follows. First, this condition implies that the area of a surface decreases as it evolves under the dynamics of (0.1). By considering, in a certain generalized sense, a surface of 'minimum area' whose boundary is C, a closed path (and therefore an invariant set) of (0.1), we find that its boundary continues to be C and that its area decreases as it evolves over a short time interval. The minimality of the original surface area is thus contradicted so that no such closed path can exist. The condition $\lambda_{n-1} + \lambda_n > 0$ similarly implies that surface areas increase in the system (0.1) and the same conclusion may be deduced from a time reversal. The result for general Lozinskiĭ norms μ is obtained in the same way by considering different measures of surface area in this argument. This method is more suitable to deal with nondissipative systems.

In this chapter, both approaches are employed to derive generalizations of Dulac criteria. In §4.1, we show that concrete conditions derived in Chapter III which imply the Hausdorff dimension of the global attractor is less than 2 will be higher

dimensional Dulac criteria for dissipative systems. The case of nondissipative systems is discussed in §4.2, where we consider more general functionals than areas for surfaces with a fixed boundary C. By examining the behaviour of these functionals under the dynamics of (0.1), we arrive at new criteria for the nonexistence of invariant closed curves for general autonomous systems. Even in the case $n = 2$ our approach yields a slightly more flexible formulation of the results of Bendixson and Dulac than the traditional ones. The same strategy is used in §4.3 to study higher dimensional invariant surfaces.

Also of interest to us here is the stability problem for periodic solutions to autonomous systems (0.1). It is well known that an appropriate notion of stability for nonconstant periodic solutions is the orbital stability. When $n = 2$, a criterion for the orbital stability of a periodic solution $x = p(t)$ to (0.1) of least period ω is given by Poincaré (see [6] or [7]) which says that $p(t)$ is asymptotically orbitally stable with asymptotic phase if $\int_0^\omega \operatorname{div} f(p(t))dt < 0$. This criterion is generalized to the case $n > 2$ in §4.4. It will be used in Chapter VII, together with the Poincaré–Bendixson property, to show the nonexistence of periodic solutions for certain concrete equations arising from Mathematical Biology.

In §4.5, ω-periodic systems are considered. Under a similar integral condition, we show that, if a weak Dulac type condition in integral form holds, the only possible periodic solutions to the periodic system are those whose period are commensurate with ω. In §4.6, as an illustration, the general theory developed here is applied to the Lorenz model. We are able to identify regions free of periodic solutions for this equation. Further applications will be seen in Chapter V and VII, where we take on the problem of global stability.

§4.1. Dissipative Systems in $\mathbf{R}^n$

We consider an autonomous system in $\mathbf{R}^n$

$$x' = f(x) \tag{1.1}$$

where $x \mapsto f(x) \in \mathbf{R}^n$ is a C^1 function defined on an open set $D \subset \mathbf{R}^n$.

We assume that (1.1) is dissipative with a bounded absorbing set $D_0 \subset D$. Then $\mathbf{A} = \omega(D_0)$ is the global attractor of (1.1) in D. Recall a result in Chapter II.

Theorem 4.1.1. *Suppose D is simply connected. If $\dim_H A < 2$, then no simple closed rectifiable curve in $\mathbf{A}$ can be invariant.*

Since a periodic trajectory gives rise to a simple closed rectifiable curve in D, condition $\dim_H \mathbf{A} < 2$ implies that (1.1) has no periodic solutions.

Corollary 4.1.2. *Under the assumptions of Theorem 4.1.1 the system (1.1) has no periodic solutions.*

In the following we will use the upper estimates for $\dim_H \mathbf{A}$ in Chapter III to derive concrete conditions which ensure $\dim_H \mathbf{A} < 2$, and thus by Theorem 4.1.1 preclude the existence of invariant simple closed rectifiable curves.

Let $|\cdot|$ denote a vector norm on $\mathbf{R}^N$, $N = \binom{n}{2}$, as well as the induced matrix norm for $N \times N$ matrices. Let μ be the corresponding Lozinskiĭ measure. We will consider a real-valued function $(x, y) \mapsto V(x, y)$ defined for $(x, y) \in D \times \mathbf{R}^N$ which is examined in Chapter III. The following result follows from Theorem 4.1.1 and the case $k = 2$ of Theorem 3.3.2 in Chapter III.

Theorem 4.1.3. *Assume that D is simply connected. Suppose there is a function $V(x, y)$ such that*

$$V(x,\, y) \geq a\,|y|, \tag{1.2}$$

$$\overset{\bullet}{V}(x,\, y) \leq -b\,|y| \tag{1.3}$$

for all $(x, y) \in \mathbf{A} \times \mathbf{R}^N$. Then no simple closed rectifiable curve in D can be invariant with respect to (1.1).

Let $x \mapsto A(x)$ be a $\binom{n}{2} \times \binom{n}{2}$ matrix-valued function that is C^1 and non-

singular in **A**. Consider the function V given by

$$V(x,\, y) = |A(x)\, y| \tag{1.4}$$

Since $A(x)$ is nonsingular and C^1 on **A**, $V(x,y)$ satisfies (1.2). From the discussion in §3.3 of Chapter III, we know that V satisfies (1.3) if the following condition holds:

$$\mu\left(A_f\, A^{-1} + A\, \frac{\partial f}{\partial x}^{[2]} A^{-1} \right) \le \delta < 0 \qquad \text{on} \quad \mathbf{A}. \tag{1.5}$$

Therefore, by Theorem 4.1.3, we have the following general criterion of Dulac type.

Theorem 4.1.4. *Suppose D is simply connected. If (1.5) is satisfied, then no simple closed rectifiable curve in D can be invariant with respect to (1.1).*

Remarks.

(i). Theorem 4.1.4 still holds if (1.5) is replaced by

$$\mu\left(-A_f\, A^{-1} - A\, \frac{\partial f}{\partial x}^{[2]} A^{-1} \right) \le \delta < 0 \qquad \text{on} \quad \mathbf{A}. \tag{1.6}$$

This can be shown using Theorem 4.1.4 and a time reversal argument.

(ii). We will see in Chapter V that, under the assumptions of Theorem 4.1.5, (1.1) can not have orbits of the following types (See Figure 4.1.1):

(1) periodic orbits;

(2) homoclinic orbits;

(3) a pair of heteroclinic orbits of the same equilibria;

(4) heteroclinic cycles,

since in each case, under the condition (1.5), the orbits give rise to simple closed rectifiable curves which are invariant with respect to (1.1). Note that this conclusion is stronger than what can be inferred from Theorem 4.1.1, since the invariant curves arising from orbits of types (2) – (4) may not be rectifiable.

Let $A = I$, we have the following result, which is first proved by J. S. Muldowney [8].

Theorem 4.1.5. *Assume that* D *is simply connected. Suppose one of the following conditions*

$$\mu\left(\frac{\partial f^{[2]}}{\partial x}\right) < 0, \qquad \mu\left(-\frac{\partial f^{[2]}}{\partial x}\right) < 0. \tag{1.7}$$

holds on the global attractor $\mathbf{A}$. *Then no simple closed rectifiable curve in* D *can be invariant with respect to* (1.1).

If the Lozinskiĭ measure in (1.7) is computed with respect to the l_1, l_∞, l_2 norms of $\mathbf{R}^N$, we arrive at the following concrete conditions each of which is Bendixson's criterion when $n = 2$.

Theorem 4.1.6 (Bendixson's Criterion in $\mathbf{R}^n$). *Suppose* D *is simply connected. Then no simple closed rectifiable curve which is invariant with respect to* (1.1) *can exist if any one of the following conditions holds on the global attractor* $\mathbf{A}$:

(i) $$\sup\left\{\frac{\partial f_r}{\partial x_r} + \frac{\partial f_s}{\partial x_s} + \sum_{q\neq r,s}\left(\left|\frac{\partial f_q}{\partial x_r}\right| + \left|\frac{\partial f_q}{\partial x_s}\right|\right) \ : 1 \le r < s \le n\right\} < 0,$$

(ii) $$\sup\left\{\frac{\partial f_r}{\partial x_r} + \frac{\partial f_s}{\partial x_s} + \sum_{q\neq r,s}\left(\left|\frac{\partial f_r}{\partial x_q}\right| + \left|\frac{\partial f_s}{\partial x_q}\right|\right) \ : 1 \le r < s \le n\right\} < 0,$$

(iii) $\lambda_1 + \lambda_2 < 0$,

(iv) $$\inf\left\{\frac{\partial f_r}{\partial x_r} + \frac{\partial f_s}{\partial x_s} - \sum_{q\neq r,s}\left(\left|\frac{\partial f_q}{\partial x_r}\right| + \left|\frac{\partial f_q}{\partial x_s}\right|\right) \ : 1 \le r < s \le n\right\} > 0,$$

(v) $$\inf\left\{\frac{\partial f_r}{\partial x_r} + \frac{\partial f_s}{\partial x_s} - \sum_{q\neq r,s}\left(\left|\frac{\partial f_r}{\partial x_q}\right| + \left|\frac{\partial f_s}{\partial x_q}\right|\right) \ : 1 \le r < s \le n\right\} > 0,$$

(vi) $\lambda_{n-1} + \lambda_n > 0$.

where $\lambda_1 \ge \cdots \ge \lambda_n$ *are eigenvalues of the symmetric matrix* $(\frac{\partial f}{\partial x}^* + \frac{\partial f}{\partial x})/2$.

Proof. If $y \in \mathbf{R}^N$ and $|y| = \sum_i |y_i|$, $\sup_i |y_i|$ or $(y^*y)^{\frac{1}{2}}$, then the Lozinskiĭ measure $\mu(\frac{\partial f}{\partial x}^{[2]})$ is the expression in (i), (ii) or (iii) and $-\mu(-\frac{\partial f}{\partial x}^{[2]})$ is the

expression in (iv), (v) or (vi), respectively.

Remark. We can see that (iii) and (vi) give Smith's conditions. Conditions (i) (ii) (iv) and (v) can be easily computed from the equations of (1.1), whereas conditions (iii) and (vi) may be difficult to calculate.

Let $x \mapsto A(x)$ be a $N \times N$ matrix-valued function, $N = \binom{n}{2}$, considered in (1.4) and (1.5). The following are special cases of the quantities $\bar{q}_k(t, K)$ and $\bar{q}_k(K)$ defined in (3.15) of Chapter III for compact invariant sets.

$$\bar{q}_2(t,\, \mathbf{A}) = \sup_{x \in \mathbf{A}} \frac{1}{t} \int_0^t \mu\big(B(x(s, x_0))\big)\, ds \tag{1.8}$$

with

$$B = A_f A^{-1} + A \frac{\partial f^{[2]}}{\partial x} A^{-1}, \tag{1.9}$$

and

$$\bar{q}_2(\mathbf{A}) = \limsup_{t \to \infty} \bar{q}_2(t,\, \mathbf{A}). \tag{1.10}$$

Theorem 3.3.6 in Chapter III when $k = 2$ shows that $\bar{q}_2(\mathbf{A}) < 0$ implies $\dim_H \mathbf{A} < 2$. We thus have the following result.

Theorem 4.1.7. *Assume that D is simply connected. If $\bar{q}_2(\mathbf{A}) < 0$, then no simple closed rectifiable curve in D can be invariant.*

Remark. The condition $\bar{q}_2(\mathbf{A}) < 0$ is less restrictive than (1.5) which requires that $\mu(B) < 0$ holds pointwisely on $\mathbf{A}$. Applications of both Theorem 4.1.4 and Theorem 4.1.7 will be seen in Chapter V and VII.

§4.2. Non-dissipative Systems.

In this section, we derive similar conditions obtained in the section §4.1 for autonomous systems which are not necessarily dissipative. Our method is to study the evolution of some general functionals, defined on surfaces with a fixed boundary

C, under the dynamics of (1.1). We show that any conditions which guarantee that such a functional decreases along the solutions of (1.1) will give rise to criteria of Bendixson or Dulac type for (1.1). In the subsection §4.2.1, the notions of rectifiable 2-surfaces and simple closed curves are defined. Then general functionals will be defined on the rectifiable 2-surfaces with a fixed simple closed rectifiable curve as their boundary. Properties of such surface functional, especially their evolution along the solutions of (1.1) are also studied. In §4.2.2, two general criteria precluding the existence of simple closed rectifiable curves invariant with respect to (1.1) are obtained based on the study of these surface functionals. Concrete Dulac type conditions derived from these criteria generalize the results in [8] and [9]. Even in the case $n = 2$ our approach yields a slightly more flexible formulation of the results of Bendixson and Dulac than the traditional ones.

§4.2.1. Evolution of Surface Functionals

We first of all recall a special case of our definition in Chapter II for normal closed surfaces. Let $U = B^2(0,1)$, the euclidean unit ball in $\mathbf{R}^2$ and let $\overline{U}$ and ∂U be its closure and boundary, respectively. If $D \subset \mathbf{R}^n$, a function $\varphi \in \operatorname{Lip}(\overline{U} \to D)$ will be described as a *simply connected rectifiable* 2*-surface in* D; a function $\psi \in \operatorname{Lip}(\partial U \to D)$ is a *closed rectifiable curve in* D and will be called *simple* if it is one-to-one. Moreover, if ψ is the restriction to ∂U of a function $\varphi : \overline{U} \to D$, we denote this by $\psi = \partial\varphi$. If D is an open, simply connected set, then

$$\Sigma(\psi, D) = \{\varphi \in \operatorname{Lip}(\overline{U} \to D) \ : \ \varphi(\partial U) = \psi(\partial U)\} \tag{2.1}$$

is nonempty for each simple closed rectifiable curve ψ in D. To see this, let (r,θ) be polar coordinates in $\mathbf{R}^2$. Since $\psi(\partial U)$ is homotopic to a point in D there is a continuous function $(r,\theta) \mapsto \widetilde{\varphi}(r,\theta) \in D$ with $\widetilde{\varphi}(r,0) = \widetilde{\varphi}(r,2\pi)$ and $\widetilde{\varphi}(1,\theta) = \psi(1,\theta)$, $0 \le r \le 1$, $0 \le \theta \le 2\pi$. To find $\varphi \in \sum(\psi, D)$, we construct a lipschitzian approximation to $\widetilde{\varphi}$ as follows. We partition $\overline{U}$ into triangular regions (a region intersecting ∂U may have a portion of ∂U as one of its sides)

and let $\varphi(u) = \widetilde{\varphi}(u) = \psi(u)$ if $u \in \partial U$ and $\varphi(u) = \widetilde{\varphi}(u)$ if u is a vertex in the interior of U; by interpolating linearly in the triangles we find $\varphi \in \mathrm{Lip}(\overline{U} \to \mathbf{R}^n)$ such that $\varphi = \partial\psi$. Moreover, since D is open, $\varphi(\overline{U}) \subset D$ by continuity if the triangular partition is fine enough so that $\varphi \in \sum(\psi, D)$.

If D_0 is the domain of f in (1.1), we consider functionals $\mathcal{S}$ on the surfaces $\mathrm{Lip}\,(\overline{U} \to D_0)$ of the form

$$\mathcal{S}\varphi = \int_{\overline{U}} S\big(\varphi(u), \tfrac{\partial}{\partial u_1}\varphi(u) \wedge \tfrac{\partial}{\partial u_2}\varphi(u)\big)\, du \tag{2.2}$$

where $u = (u_1, u_2)$ and $(x,y) \mapsto S(x,y)$ is a real-valued function with $x \in D_0$, $y \in \mathbf{R}^N$, $N = \binom{n}{2}$. We require also that S be locally Lipschitzian on its domain and that $\lim_{h\to 0+} \frac{1}{h}[S(x+ha, y+hb) - S(x,y)]$ exists for all $(x,y) \in D \times \mathbf{R}^N$ and all $(a,b) \in \mathbf{R}^n \times \mathbf{R}^N$. The integral in (2.2) exists since the partial derivatives of φ exist almost everywhere in $\overline{U}$ and are bounded by the lipschitz constant of φ.

We define $\overset{\bullet}{S}$ by

$$\overset{\bullet}{S}(x,y) = \lim_{h\to 0+} \frac{1}{h}\left[S\left(x + hf(x), y + h\,\frac{\partial f}{\partial x}^{[2]}(x)y\right) - S(x,y)\right]. \tag{2.3}$$

Thus $\overset{\bullet}{S} = \frac{\partial S}{\partial x}^{*} f + \frac{\partial S}{\partial y}^{*}\frac{\partial f}{\partial x}^{[2]} y$ almost everywhere, since S is Lipschitzian and therefore differentiable almost everywhere. When $n = 2$, $\overset{\bullet}{S} = \frac{\partial S}{\partial x}^{*} f + \frac{\partial S}{\partial y}(\mathrm{div} f)y$.

Proposition 4.2.1. *Suppose* $\varphi_0 \in \mathrm{Lip}\,(\overline{U} \to \mathbf{R}^n)$ *and* $\varphi_t(u) = x(t, \varphi_0(u))$. *Then* $\varphi_t \in \mathrm{Lip}\,(\overline{U} \to \mathbf{R}^n)$, *the right-hand derivative* $D_t^+\mathcal{S}\varphi_t$ *exists and*

$$D_t^+\mathcal{S}\varphi_t = \int_{\overline{U}} \overset{\bullet}{S}(\varphi_t, \tfrac{\partial}{\partial u_1}\varphi_t \wedge \tfrac{\partial}{\partial u_2}\varphi_t) \tag{2.4}$$

as long as $\varphi_t(u)$ *exists for each* $u \in \overline{U}$.

Proof. For each $u \in \overline{U}$, $\varphi_t(u)$ is a solution of (1.1). Therefore $z_i(t) = \frac{\partial}{\partial u_i}\varphi_t(u) = \frac{\partial x}{\partial x_0}(t, \varphi_0(u))\frac{\partial}{\partial u_i}\varphi_0(u)$ satisfies $\frac{dz_i}{dt} = \frac{\partial f}{\partial x}(\varphi_t(u))z_i$, $i = 1,2$ and $y(t) = \frac{\partial}{\partial u_1}\varphi_t(u) \wedge \frac{\partial}{\partial u_2}\varphi_t(u)$ satisfies $\frac{dy}{dt} = \frac{\partial f}{\partial x}^{[2]}(\varphi_t(u))y$. It follows that

$$\lim_{h\to 0+} \frac{1}{h}\left[S\left(\varphi_{t+h}, \tfrac{\partial}{\partial u_1}\varphi_{t+h} \wedge \tfrac{\partial}{\partial u_2}\varphi_{t+h}\right) - S\left(\varphi_t, \tfrac{\partial}{\partial u_1}\varphi_t \wedge \tfrac{\partial}{\partial u_2}\varphi_t\right)\right]$$

exists and equals $\overset{\bullet}{S}(\varphi_t, \frac{\partial}{\partial u_1}\varphi_t \wedge \frac{\partial}{\partial u_2}\varphi_t)$. From this and the Lebesgue Dominated Convergence Theorem we deduce Proposition 4.2.1.

The surface area, counting multiplicities, of $\varphi_t(\overline{U})$ is $\mathcal{S}\varphi_t = \int_{\overline{U}} |\frac{\partial}{\partial u_1}\varphi_t \wedge \frac{\partial}{\partial u_2}\varphi_t|$. Here $S(x,y) = |y| = (y^*y)^{1/2}$ is the ℓ^2 norm of $y \in \mathbf{R}^N$. For this functional

$$\overset{\bullet}{S}(x,y) = S(x,y)^{-1}\,\frac{1}{2}\,y^*\left(\frac{\partial f}{\partial x}^* + \frac{\partial f}{\partial x}\right)^{[2]} y$$

and therefore

$$(\lambda_{n-1}(x) + \lambda_n(x))\,S(x,y) \le \overset{\bullet}{S}(x,y) \le (\lambda_1(x) + \lambda_2(x))\,S(x,y) \tag{2.5}$$

where $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$ are the eigenvalues of $\frac{1}{2}(\frac{\partial f}{\partial x}^* + \frac{\partial f}{\partial x})$, since the eigenvalues of $\frac{1}{2}(\frac{\partial f}{\partial x}^* + \frac{\partial f}{\partial x})^{[2]}$ are $\lambda_i + \lambda_j$, $1 \le i < j \le n$ (see Appendix B, Proposition 2.6). Thus the surface area of $\varphi_t(\overline{U})$ increases (decreases) in t as long as $\varphi_t(\overline{U})$ lies in a set where $\lambda_{n-1} + \lambda_n > 0$ $(\lambda_1 + \lambda_2 < 0)$.

If $x \mapsto H(x)$ is a C^1 real symmetric $\binom{n}{2} \times \binom{n}{2}$ matrix-valued function and $S(x,y) = y^*H(x)y$, then

$$\overset{\bullet}{S}(x,y) = y^*\left(H_f + \frac{\partial f}{\partial x}^{[2]*} H + H\frac{\partial f}{\partial x}^{[2]}\right) y, \tag{2.6}$$

where H_f is the matrix obtained by replacing each entry h_{ij} of H by $(h_{ij})_f = \frac{\partial h_{ij}}{\partial x}^* f$, its directional derivative in the direction f. In this case $\mathcal{S}\varphi_t$ increases (decreases) if $\varphi_t(\overline{U})$ lies in a set where $H_f + \frac{\partial f}{\partial x}^{[2]*} H + H\frac{\partial f}{\partial x}^{[2]}$ is positive (negative) definite.

A general class of functionals S in which we are interested is given by $S(x,y) = |A(x)y|$, where $|\cdot|$ is any norm on $\mathbf{R}^N$, $N = \binom{n}{2}$, and $x \mapsto A(x)$ is a C^1 nonsingular real $N \times N$ matrix-valued function. In this case, it follows from (2.3) that

$$-\mu(-B)\,S \le \overset{\bullet}{S} \le \mu(B)\,S, \tag{2.7}$$

where $B = A_f A^{-1} + A\,\frac{\partial f}{\partial x}^{[2]}\,A^{-1}$ and μ is the Lozinskiĭ measure corresponding to $|\cdot|$ When $A = I$, $B = \frac{\partial f}{\partial x}^{[2]}$ and $\mu(B) = \lambda_1 + \lambda_2$, $-\mu(-B) = \lambda_{n-1} + \lambda_n$ in

the case that $|y| = (y^*y)^{\frac{1}{2}}$. When $n = 2$, $-\mu(-B) = \mu(B) = \operatorname{div} f$ so that (2.7) gives $\dot{S} = (\operatorname{div} f)S$, the familiar formula of Liouville and Jacobi ([7], page 44).

We now consider functionals $\mathcal{A}$ on $\mathrm{Lip}(\overline{U} \to \mathbf{R}^n)$ defined by

$$\mathcal{A}\varphi = \int_U \left| \frac{\partial \varphi}{\partial u_1} \wedge \frac{\partial \varphi}{\partial u_2} \right|^p \tag{2.8}$$

where $|\cdot|$ is any norm on $\mathbf{R}^N$ and $p \geq 1$. For example, if $|y| = (y^*y)^{\frac{1}{2}}$ and $p = 1$, $\mathcal{A}$ is the surface area of $\varphi(\overline{U})$. We show that, if $\partial\varphi$ is a simple closed curve, then $\mathcal{A}\varphi$ has a positive lower bound which depends only on $|\cdot|$, p and $\partial\varphi$.

Proposition 4.2.2. *Suppose ψ is a simple closed rectifiable curve in $\mathbf{R}^n$. Then there exists $\delta > 0$ such that*

$$\mathcal{A}\varphi \geq \delta$$

for all $\varphi \in \Sigma(\psi, \mathbf{R}^n)$.

Proof. Since all norms in $\mathbf{R}^N$ are equivalent, it is sufficient to prove the proposition in the case that $|y| = (y^*y)^{1/2}$. It also suffices to prove the statement for $\varphi \in \Sigma(\psi, K)$, where K is the convex hull of $\psi(\partial U)$. This follows from the fact that if Π is any $(n-1)$-dimensional hyperplane in $\mathbf{R}^n$ which does not intersect $\psi(\partial U)$ and $\varphi \in \Sigma(\psi, \mathbf{R}^n)$, by orthogonal projection onto Π, if necessary, we can find $\widetilde{\varphi} \in \Sigma(\psi, \mathbf{R}^n)$ such that $\mathcal{A}\widetilde{\varphi} \leq \mathcal{A}\varphi$ and $\widetilde{\varphi}(\overline{U})$ does not cross Π. Next, observe that $\int_0^{2\pi} |\psi'|^2 > 0$, where $\psi(\theta) = \psi(\cos\theta, \sin\theta)$, since ψ is one-to-one. Choose the continuous function b from ∂U to $\mathbf{R}^n$ such that, if $b(\theta) = (b \circ \psi)(\theta)$, $\int_0^{2\pi} b^*\psi'$ is sufficiently close to $\int_0^{2\pi} |\psi'|^2$ to ensure $\int_0^{2\pi} b^*\psi' > 0$. Then the function b may be extended continuously to $\mathbf{R}^n$ and b may be approximated on $\mathbf{R}^n$ by a C^1 function a such that $\alpha\varphi(\partial U) = \alpha\psi(\partial U)$ is sufficiently close to $\int_0^{2\pi} b^*\psi'$ to ensure $\alpha\varphi(\partial U) > 0$, where α is the 1-form defined by

$\alpha = \sum_i a_i(x)\,dx_i$. But Stokes' Theorem implies

$$\begin{aligned}\alpha\varphi(\partial U) &= d\alpha\,\varphi(\overline{U})\\ &= \int_{\varphi(\overline{U})} \sum_{i<j} \left(\frac{\partial a_j}{\partial x_i} - \frac{\partial a_i}{\partial x_j}\right) dx_i \wedge dx_j\\ &= \int_{\overline{U}} z^*(u)\,y(u)\,du,\end{aligned}$$

where $y(u) = \frac{\partial}{\partial u_1}\varphi(u) \wedge \frac{\partial}{\partial u_2}\varphi(u)$, $z_i(u) = \frac{\partial a_{i_2}}{\partial x_{i_1}}(x) - \frac{\partial a_{i_1}}{\partial x_{i_2}}(x)$, $x = \varphi(u)$, $(i) = (i_1, i_2)$, $i = 1, \ldots, N = \binom{n}{2}$. Since a is C^1 on $\mathbf{R}^n$ and $\varphi(u) \in K$, there is a constant M independent of φ such that $|z(u)| \leq M$ for all $u \in \overline{U}$. Thus Hölder's Inequality implies

$$\begin{aligned}0 < \alpha\psi(\partial U) = \alpha\varphi(\partial U) &\leq \int_{\overline{U}} |z|\,|y|\\ &\leq \left(\int_{\overline{U}} |z|^q\right)^{\frac{1}{q}} \left(\int_{\overline{U}} |y|^p\right)^{\frac{1}{p}}\\ &\leq \pi^{\frac{1}{q}} M (\mathcal{A}\varphi)^{\frac{1}{p}}.\end{aligned}$$

We conclude that Proposition 4.2.2 holds with $\delta = [\alpha\psi(\partial U)/\pi^{\frac{1}{q}} M]^p$. □

A functional $\mathcal{S}$ of the form (2.2) is said to be *strongly decreasing* with respect to (1.1) on $D \subset \mathbf{R}^n$ if there exist constants p, a, b with $p \geq 1$, $a \geq 0$, $b \geq 0$ and $a + b > 0$ such that

$$\overset{\bullet}{S}(x, y) \leq -(a + b|y|^p) \tag{2.9}$$

if $x \in D$ and $y \in \mathbf{R}^N$.

It follows from Propositions 4.2.1, 4.2.2 that

$$D_t^+ \mathcal{S}\varphi_t \leq -(a\pi + b\delta) \tag{2.10}$$

if $\varphi_t \in \Sigma(\psi, D)$ and D is a set where (2.9) holds.

§4.2.2. Bendixson's Criterion

A subset B of D is *invariant* with respect to (1.1) if $x(t,B) = B$ for all $t \in (-\infty,\infty)$. A simple closed rectifiable curve ψ in D is *invariant with respect to* (1.1) if $\psi(\partial U)$ is invariant with respect to (1.1).

Criterion I. *Suppose that ψ is a simple closed rectifiable curve in D which is invariant with respect to* (1.1). *Then there cannot exist a functional $\mathcal{S}$ of the form* (2.2) *such that (a) and (b) are satisfied:*

(a) $-\infty < m = \inf\{\mathcal{S}\varphi : \varphi \in \Sigma(\psi, D)\}$.

(b) *There is a sequence of surfaces $\varphi^k \in \Sigma(\psi, D)$ such that $m = \lim_{k\to\infty} \mathcal{S}\varphi^k$ and $\mathcal{S}$ is strongly decreasing with respect to* (1.1) *on $\{x(t, \varphi^k(\overline{U})) : t \in [0,\varepsilon],\ k = 1,2,\dots\}$ for some $\varepsilon > 0$.*

Criterion II. *Suppose that ψ is a simple closed rectifiable curve in D which is invariant with respect to* (1.1). *Then there cannot exist a functional $\mathcal{S}$ such that (a) and (b) are satisfied:*

(a) $-\infty < m = \inf\{\mathcal{S}\varphi : \varphi \in \Sigma(\psi, D)\}$.

(b) *There is a surface $\varphi_0 \in \Sigma(\psi, D)$ such that $\mathcal{S}$ is strongly decreasing with respect to* (1.1) *on $\{x(t, \varphi_0(\overline{U})) : t \in [R,\infty)\}$ for some $R > 0$.*

To establish these criteria, observe that the invariance of ψ implies $\varphi_t \in \Sigma(\psi, D)$ if $\varphi_0 \in \Sigma(\psi, D)$ as long as $\varphi_t(u)$ exists for each $u \in \overline{U}$. From Proposition 4.2.1 and (2.10) the conditions of Criterion I imply $\mathcal{S}\varphi^k_\varepsilon \le \mathcal{S}\varphi^k - (a\pi + b\delta)\varepsilon$, where $\varphi^k_t(u) = x(t, \varphi^k(u))$, $k = 1,2,\dots$. Since $\varphi^k \in \Sigma(\psi, D)$ implies $\varphi^k_\varepsilon \in \Sigma(\psi, D)$ and $\limsup_{k\to\infty} \mathcal{S}\varphi^k_\varepsilon \le m - (a\pi + b\pi)\varepsilon < m$ we have a contradiction of (a).

Similarly, the conditions of Criterion II imply $\mathcal{S}\varphi_t \le \mathcal{S}\varphi_R - (a\pi + b\delta)(t - R)$, $R \le t < \infty$, so that $\varphi_t \in \Sigma(\psi, D)$ and $\lim_{t\to\infty} \mathcal{S}\varphi_t = -\infty < m$, again contradicting (a).

We now deduce more concrete expressions from these criteria, beginning with six conditions in $\mathbf{R}^n$ each of which is the Bendixson's criterion when $n = 2$.

Theorem 4.2.3 (Bendixson's Criterion in $\mathbf{R}^n$). *A simple closed rectifiable curve which is invariant with respect to* (1.1) *cannot exist if any one of the following conditions is satisfied on* $\mathbf{R}^n$:

$$\text{(i)}\quad \sup\left\{\frac{\partial f_r}{\partial x_r}+\frac{\partial f_s}{\partial x_s}+\sum_{q\neq r,s}\left(\left|\frac{\partial f_q}{\partial x_r}\right|+\left|\frac{\partial f_q}{\partial x_s}\right|\right)\ :\ 1\le r<s\le n\right\}<0,$$

$$\text{(ii)}\quad \sup\left\{\frac{\partial f_r}{\partial x_r}+\frac{\partial f_s}{\partial x_s}+\sum_{q\neq r,s}\left(\left|\frac{\partial f_r}{\partial x_q}\right|+\left|\frac{\partial f_s}{\partial x_q}\right|\right)\ :\ 1\le r<s\le n\right\}<0,$$

$$\text{(iii)}\quad \lambda_1+\lambda_2<0,$$

$$\text{(iv)}\quad \inf\left\{\frac{\partial f_r}{\partial x_r}+\frac{\partial f_s}{\partial x_s}-\sum_{q\neq r,s}\left(\left|\frac{\partial f_q}{\partial x_r}\right|+\left|\frac{\partial f_q}{\partial x_s}\right|\right)\ :\ 1\le r<s\le n\right\}>0,$$

$$\text{(v)}\quad \inf\left\{\frac{\partial f_r}{\partial x_r}+\frac{\partial f_s}{\partial x_s}-\sum_{q\neq r,s}\left(\left|\frac{\partial f_r}{\partial x_q}\right|+\left|\frac{\partial f_s}{\partial x_q}\right|\right)\ :\ 1\le r<s\le n\right\}>0,$$

$$\text{(vi)}\quad \lambda_{n-1}+\lambda_n>0.$$

where $\lambda_1\ge\cdots\ge\lambda_n$ *are eigenvalues of* $\frac{1}{2}(\frac{\partial f}{\partial x}^*+\frac{\partial f}{\partial x})$.

Proof. This result may be deduced from either of the preceding criteria. However, since solutions of (1.1) do not necessarily exist globally, the technicalities in using Criterion I are fewer. If $y\in\mathbf{R}^N$ and $|y|=\sup_i|y_i|$, $\sum_i|y_i|$ or $(y^*y)^{\frac{1}{2}}$, then the Lozinskiǐ measure $\mu(\frac{\partial f}{\partial x}^{[2]})$ is the expression in (i), (ii) or (iii) and $-\mu(-\frac{\partial f}{\partial x}^{[2]})$ is the expression in (iv), (v) or (vi), respectively. It follows from (2.7) with $A=I$ that, if $S(x,y)=|y|$, $\mathcal{S}$ is strongly decreasing with respect to (1.1) on any compact subset of $\mathbf{R}^n$ if the corresponding condition (i), (ii) or (iii) holds and that $\mathcal{S}$ is strongly decreasing on compacta with respect to (1.1) with reversed time if (iv), (v) or (vi) holds. Since $\mathcal{S}\varphi\ge 0$ for every φ, it remains only to show that, if

ψ is a simple closed curve, there is a sequence in $\Sigma(\psi, D)$ which minimizes $\mathcal{S}$ over $\Sigma(\psi, \mathbf{R}^n)$, where $D = \{x : |x_i| \leq c_i\}$ satisfies $\psi(\partial U) \subset D$. This follows from the observation that, if $\varphi \in \Sigma(\psi, \mathbf{R}^n)$ and $\widetilde{\varphi}_i(u) = -c_i, \quad \varphi_i(u), \quad c_i$ as $\varphi_i(u) \in (-\infty, -c_i], \quad (-c_i, c_i), \quad [c_i, \infty), \quad i = 1, \ldots, n,$ then $\widetilde{\varphi} \in \Sigma(\psi, D)$ and $\mathcal{S}\widetilde{\varphi} \leq \mathcal{S}\varphi$, since $|\frac{\partial}{\partial u_1}\widetilde{\varphi} \wedge \frac{\partial}{\partial u_2}\widetilde{\varphi}| \leq |\frac{\partial}{\partial u_1}\varphi \wedge \frac{\partial}{\partial u_2}\varphi|$. $\square$

Any of the criteria (i), ... ,(vi) of Theorem 3.3 may be modified, following Smith [9], by replacing f by αf, where $x \mapsto \alpha(x)$ is a positive C^1 scalar-valued function. This amounts to a change of the independent variable t and gives a generalization of Theorem 2.3 in the spirit of Dulac (see [8], Remark (d)). This arbitrary function introduced into the criteria may be replaced by $\binom{n}{2}^2$ such functions if we consider the functional $\mathcal{S}$ defined by (2.3) with $S(x, y) = |A(x)y|$, where $x \mapsto A(x)$ is a C^1 nonsingular $\binom{n}{2} \times \binom{n}{2}$ matrix-valued function on D. Here $\mathcal{S}\varphi = \int_{\overline{U}} |A(\varphi)\frac{\partial}{\partial u_1}, \varphi \wedge \frac{\partial}{\partial u_2}, \varphi|$. It follows from (2.7) that $\overset{\bullet}{S}(x, y) < 0$ whenever $\mu(B) < 0$ where μ is the Lozinskiĭ measure corresponding to $|\cdot|$ and that S is strongly decreasing on sets where $\mu(B) \leq -b < 0$. Similarly $\overset{\bullet}{S}(x, y) > 0$ when $-\mu(-B) > 0$.

We will say that D has the *minimum property with respect to* S if, for each simple closed rectifiable curve ψ in D, there is a minimizing sequence $\varphi^k \in \Sigma(\psi, D)$ for $\mathcal{S}$ such that $\cup_k \varphi^k(\overline{U})$ has compact closure in D. It follows that, if D has this property and $\mu(B) < 0$, then $\varphi^k_t(u)$ exists for each $u \in \overline{U}$ and $k = 1, 2, \ldots, \quad 0 \leq t \leq \varepsilon$, for some $\varepsilon > 0$, and that the conditions (a) and (b) of Criterion I are satisfied for each simple closed rectifiable curve ψ in D. Thus we have the following theorem.

Theorem 4.2.4. *Suppose that*

(a) D *has the minimum property with respect to* $S(x, y) = |A(x)y|$.

(b) $\mu\left(A_f A^{-1} + A\dfrac{\partial f}{\partial x}^{[2]} A^{-1}\right) < 0$ *on* D.

Then no simple closed rectifiable curve in D *is invariant with respect to* (1.1).

Remark. The condition (b) may be replaced by $\mu(-A_f A^{-1} - A\frac{\partial f}{\partial x}^{[2]} A^{-1}) < 0$

by using a time reversal argument.

With $A = I$ and $D = \mathbf{R}^n$, we obtain Theorem 2.3 if $|\cdot|$ is any of the three norms mentioned in the proof of that theorem. In fact the proof is simply a demonstration that $\mathbf{R}^n$ has the minimum property with respect to $S(x, y) = |y|$. The same argument applies equally well to any absolute norm, where $|\cdot|$ is said to be *absolute* if $|y|$ is unchanged by replacing the components y_i of y by $|y_i|$.

Theorem 4.2.5. *Suppose that one of*

$$\mu\left(\frac{\partial f}{\partial x}^{[2]}\right) < 0, \qquad \mu\left(-\frac{\partial f}{\partial x}^{[2]}\right) < 0$$

holds on $\mathbf{R}^n$ *where* μ *is a Lozinskiĭ measure corresponding to an absolute norm* $|\cdot|$ *on* $\mathbf{R}^N$, $N = \binom{n}{2}$. *Then no simple closed rectifiable curve in* $\mathbf{R}^n$ *is invariant with respect to* (1.1).

The criteria for dissipative systems obtained in §4.1 can be shown to follow from these two criteria as well. As a matter of fact, it follows from Criterion II that if S satisfies (a) and (1.1) has an absorbing set D_0 on which S is strongly decreasing, then no simple closed curve ψ in D for which $\Sigma(\psi, D)$ is nonempty can be invariant with respect to (1.1). This gives the following result.

Theorem 4.2.6. *Suppose that*

(*a*) D *is simply connected.*

(*b*) $\mu(A_f A^{-1} + A\frac{\partial f}{\partial x}^{[2]}A^{-1}) \leq -\delta < 0$ *on a set* D_0 *which is absorbing with respect to* (1.1).

Then there is no simple closed rectifiable curve in D *which is invariant with respect to* (1.1).

Remarks.

(i). If $S(x, y) = \alpha(x)$, where α is C^1, then $\dot{S} = \frac{\partial \alpha}{\partial x}^{*} f$ and S is strongly decreasing on any set where $\frac{\partial \alpha}{\partial x}^{*} f \leq -\delta < 0$. Our criteria then translate into

a weak form of the familiar observation that no nonconstant periodic solution can exist if there is a real-valued function α which decreases along trajectories of (1.1).

(ii). Finally we observe that, when $n = 2$, Dulac's criterion follows from consideration of $S(x,y) = \alpha(x)y$. Then, for any simple closed curve ψ with $\varphi \in \Sigma(\psi, D)$, $\mathcal{S}\varphi = \int_{D_1} \alpha$, where D_1 is the region bounded by $\psi(\partial U)$, and so $\mathcal{S}\varphi$ depends only on ψ. If ψ is invariant and $D_1 \subset D$, we may find a function $\varphi \in \Sigma(\psi, D_1)$ such that $\frac{\partial \varphi}{\partial u_1} \wedge \frac{\partial \varphi}{\partial u_2} = \frac{\partial(\varphi_1, \varphi_2)}{\partial(u_1, u_2)} \geq 0$ and the constant sequence $\varphi^k = \varphi$, $k = 1, 2, \ldots$ is minimizing for $\mathcal{S}$. Also $\mathcal{S}$ is strongly decreasing on D_1 if $\frac{\partial \alpha}{\partial x}^* f + \alpha \operatorname{div} f < 0$. This is Dulac's condition. Criterion I and II now both imply Dulac's criterion. Here it was not necessary to assume $\alpha(x)$ was of one sign. In contrast, the higher dimensional analogue needed to consider $S(x,y) = |A(x)y|$ where $A(x)$ is nonsingular. This condition may be relaxed somewhat if a more general definition of 'strongly decreasing' is given, replacing the constants a, b of (2.9) by functions $a(x)$, $b(x)$ in which case one also needs an extension of Proposition 4.2.2 to a functional $\mathcal{A} = \mathcal{S}$ determined by (2.2) with $S(x,y) = b(x)|y|^p$.

§4.3. Higher Dimensional Invariant Structures

In this section, the ideas in the first two sections are further explored so that we can derive conditions to preclude higher dimensional structures invariant with respect to (1.1).

First of all, we assume (1.1) is dissipative with a bounded absorbing set D_0 and the global attractor $\mathbf{A}$. Let $U \subset \mathbf{R}^{k+1}$ be a bounded connected open set with boundary ∂U and closure $\bar{U}$. For an integer $k \geq 0$, recall that a $(k+1)$-surface in D is a mapping $\varphi \in C(\bar{U} \to D)$ and a normal k-boundary in D is a mapping $\psi \in C(\partial U \to D)$ which is one to one and Lipschitz continuous on ∂U. D is called boundedly k-connected if, for each compact subset K of D and any family of k-boundary $\{\psi_\alpha\}_{\alpha \in \Lambda}$ in K, there exists a bounded set B such that $K \subset B \subset D$ and each ψ_α bounds a $(k+1)$-surface in B. The following results are established

in Chapter II.

Theorem 4.3.1. *If* $\dim_H \mathbf{A} < k+1$, *then there can be no invariant normal k-boundary in D which bounds a $(k+1)$-surface in D.*

Theorem 4.3.2. *Suppose that D is boundedly k-connected. If $\dim_H \mathbf{A} < k+1$, then no normal k-boundary in D can be invariant with respect to (1.1).*

Using the upper estimates for $\dim_H \mathbf{A}$ developed in Chapter III, we can derive concrete criteria which preclude the existence of invariant normal k-boundary.

Let $(x,y) \mapsto V(x,y)$ be the real-valued function wich is Lipschitz continuous on $D \times \mathbf{R}^N$, $N = \binom{n}{k}$ considered in §3.3 of Chapter III, and define

$$\overset{\bullet}{V}(x,y) = \lim_{h \to 0+} \frac{1}{h} \left[V\left(x + hf(x),\ y + h\,\frac{\partial f^{[k]}}{\partial x}(x)y\right) - V(x,y) \right] \tag{3.1}$$

for all $(x,y) \in D \times \mathbf{R}^N$ and all $(a,b) \in \mathbf{R}^n \times \mathbf{R}^N$, as in (3.3) of Chapter III. Thus

$$\overset{\bullet}{V}(x,y) = \frac{\partial V^*}{\partial x} f(x) + \frac{\partial V^*}{\partial y}\,\frac{\partial f^{[k]}}{\partial x}\, y \tag{3.2}$$

almost everywhere.

The following result follows from Theorem 3.3.2 in Chapter III.

Theorem 4.3.3. *Assume that D is boundedly k-connected . Suppose that there exists constants $a,\ b > 0$ and a function $V(x,y)$ such that, for a vector norm $|\cdot|$ in $\mathbf{R}^N$, $N = \binom{n}{k}$,*

(a) $V(x,y) \geq a\,|y|$,

(b) $\overset{\bullet}{V}(x,y) \leq -b\,|y|$,

for all x in the global attractor $\mathbf{A}$, and $y \in \mathbf{R}^N$. Then no normal k-boundary in D can be invariant with respect to (1.1).

Consider a general function $V(x,y)$ given by

$$V(x,y) = |A(x)\,y| \tag{3.3}$$

where $x \mapsto A(x)$ is a $N \times N$ matrix-valued function C^1 and nonsingular for all $x \in \mathbf{A}$. Then from Chapter III we know that

$$V(x,y) \geq c\,|y|$$

for all $x \in \mathbf{A}$ and $y \in \mathbf{R}^N$, and the constant c is independent of x, y, and

$$-\mu(-B)V \leq \overset{\bullet}{V} \leq \mu(B)V \tag{3.4}$$

with

$$B = A_f A^{-1} + A\,\frac{\partial f^{[k]}}{\partial x}\,A^{-1} \tag{3.5}$$

and μ is the Lozinskiĭ measure corresponding to the vector norm $|\cdot|$. The following results follows from Theorem 3.3.3 of Chapter III.

Theorem 4.3.4. *Suppose that D is boundedly k-connected. If*

$$\mu(B) < 0 \qquad \text{on} \quad \mathbf{A}, \tag{3.6}$$

then no normal k-boundary in D can be invariant with respect to (1.1).

Remark. Theorem 4.3.4 still holds if (3.6) is replaced by the following condition

$$\mu(-B) < 0 \qquad \text{on} \quad \mathbf{A}. \tag{3.7}$$

when $A = I$, Theorem 4.3.4 yields the following corollary.

Theorem 4.3.5. *Assume that D is boundedly k-connected. Suppose that one of*

$$\mu\left(\frac{\partial f^{[k]}}{\partial x}\right) < 0, \qquad \mu\left(-\frac{\partial f^{[k]}}{\partial x}\right) < 0 \tag{3.8}$$

holds on $\mathbf{A}$. Then no normal k-boundary in D can be invariant with respect to (1.1).

Next, we will develop similar results when (1.1) is not assumed to be dissipative.

We will consider a general class of functionals $\mathcal{S}$ defined on a family of k-dimensional surfaces in $\mathbf{R}^n$. We will say that a Lipschitzian function $u \mapsto \varphi(u)$ from $\mathbf{R}^k$ to $\mathbf{R}^n$ is a *rectifiable k-surface in $\mathbf{R}^n$* if its domain is $\overline{U}$, the closure of a nonempty connected open set $U \subset \mathbf{R}^k$ to which Stokes' Theorem $\omega(\partial U) = d\omega(U)$ for C^1 $(k-1)$-forms ω is applicable. The restriction $\psi = \partial\varphi$ of such a surface to the boundary ∂U of its domain is a *normal $(k-1)$-boundary* if φ is one-to-one on ∂U. Note that $\psi(\partial U)$ might have more than one connected component but that self-intersections do not occur. We consider the functional $\varphi \mapsto \mathcal{S}\varphi$ defined by

$$\mathcal{S}\varphi = \int_{U} \left| A(\varphi) \frac{\partial \varphi}{\partial u_1} \wedge \cdots \wedge \frac{\partial \varphi}{\partial u_k} \right| \tag{3.9}$$

where $|\cdot|$ is a norm on $\mathbf{R}^{\binom{n}{k}}$, $x \mapsto A(x)$ is a real C^1 nonsingular $\binom{n}{k} \times \binom{n}{k}$ matrix-valued function and $u = (u_1, \ldots, u_k)$. In the case that $|\cdot|$ is the ℓ^2 norm and $A = I$ we denote the functional defined by (3.9) by $\mathcal{A}$. Note that $\mathcal{A}\varphi$ is the k-dimensional volume of $\varphi(\overline{U})$ counting multiplicities and that, in particular (see [5], page 25),

$$\mathcal{A}\varphi \geq \mathcal{H}^k \varphi(\overline{U}) \tag{3.10}$$

where $\mathcal{H}^k$ is the k-dimensional Hausdorff measure.

Let $x(t)$ be the solution to (1.1). The linear variational equation of (1.1) with respect to $x(t)$ is given by

$$y'(t) = \frac{\partial f}{\partial x}(x(t))\, y(t). \tag{3.11}$$

Suppose $y^1(t), \ldots, y^k(t)$ are solutions of (3.11), then $\mathbf{R}^{\binom{n}{k}}$-valued function $z(t) = y^1(t) \wedge \cdots \wedge y^k(t)$ is a solution of the k-th compound equation of (3.11) (see Appendix B, Theorem 3.1)

$$z'(t) = \frac{\partial f}{\partial x}^{[k]}(x(t))\, z(t). \tag{3.12}$$

If φ_0 is a rectifiable k-surface in $\mathbf{R}^n$, let the k-surface φ_t be defined by $\varphi_t(\cdot) = x(t, \varphi_0(\cdot))$. Since $y(t) = \frac{\partial \varphi_t}{\partial u_i}$ satisfies (3.11) with $x(t) = \varphi_t$, it follows

that $z(t) = \frac{\partial\varphi_t}{\partial u_1} \wedge \cdots \wedge \frac{\partial\varphi_t}{\partial u_k}$ satisfies (3.12) and that $w(t) = A(\varphi_t)\frac{\partial\varphi_t}{\partial u_1} \wedge \cdots \wedge \frac{\partial\varphi_t}{\partial u_k}$ is a solution of

$$w'(t) = B(\varphi_t)w(t) \tag{3.13}$$

where

$$B = A_f A^{-1} + A\frac{\partial f^{[k]}}{\partial x} A^{-1} \tag{3.14}$$

and $A_f(x)$ is the matrix obtained by replacing each entry in $A(x)$ by its directional derivative in the direction $f(x)$. It follows that

$$D_t^+ \mathcal{S}\varphi_t \le \int_{\overline{U}} \mu(B(\varphi_t)) \left| A(\varphi_t)\frac{\partial\varphi_t}{\partial u_1} \wedge \cdots \wedge \frac{\partial\varphi_t}{\partial u_k}\right| \tag{3.15}$$

as long as $\varphi_t(\overline{U})$ is in the domain of f. Here $\mu(B)$ is the Lozinskiĭ measure of the matrix B corresponding to the norm $|\cdot|$. The inequality (3.15) follows from the fact that $D_t|w(t)| \le \mu\big(B(\varphi_t)\big)|w(t)|$ (see Appendix A, Theorem 3.1).

We investigate the implications of the condition

$$\mu(B) < 0 \tag{3.16}$$

holding in a subset of $\mathbf{R}^n$. When $A(x) = \alpha(x)I$, $\alpha(x) > 0$ and $k = n$, $\mu(B) = \frac{1}{\alpha}\operatorname{div}(\alpha f)$ so that (3.16) is Dulac's condition. When $A(x) = I$ and $|\cdot|$ is the l_1, l_∞, l_2 norm, then

$$\mu(B) = \begin{cases} \sup_{(j)} \left[\dfrac{\partial f_{j_1}}{\partial x_{j_1}} + \cdots + \dfrac{\partial f_{j_k}}{\partial x_{j_k}} + \sum\limits_{i\notin(j)}\left(\left|\dfrac{\partial f_i}{\partial x_{j_1}}\right| + \cdots + \left|\dfrac{\partial f_i}{\partial x_{j_k}}\right|\right)\right], \\[2ex] \sup_{(i)} \left[\dfrac{\partial f_{i_1}}{\partial x_{i_1}} + \cdots + \dfrac{\partial f_{i_k}}{\partial x_{i_k}} + \sum\limits_{j\notin(i)}\left(\left|\dfrac{\partial f_{i_1}}{\partial x_j}\right| + \cdots + \left|\dfrac{\partial f_{i_k}}{\partial x_j}\right|\right)\right], \\[2ex] \lambda_1 + \cdots + \lambda_k. \end{cases} \tag{3.17}$$

Thus, when $k = 2$, $A(x) = I$ and $|\cdot|_k$ is the ℓ^2 norm, (3.16) is Smith's condition $\lambda_1 + \lambda_2 < 0$.

If φ_0 is a rectifiable k-surface in $\mathbf{R}^n$ such that $\varphi_t(\overline{U})$ is a subset of a compact region D where (3.16) holds, then (3.15) implies that $D_t^+\mathcal{S}\varphi_t \le -\eta\mathcal{S}\varphi_t$ where η is a positive constant. Thus $\lim_{t\to\infty}\mathcal{S}\varphi_t = 0$. Since A is nonsingular,

(3.10) implies $\mathcal{S}\varphi_t \geq \gamma \mathcal{A}\varphi_t \geq \gamma \mathcal{H}^k \varphi_t(\overline{U})$, where γ is a positive constant. Therefore, $\lim_{t\to\infty} \mathcal{H}^k \varphi_t(\overline{U}) = 0$. In particular, we have the following theorem which gives the result of Butler et al. [2] when $k = n$. Recall that a set $K \subset \mathbf{R}^n$ is *invariant* (*positively invariant*) with respect to (1.1) if $\varphi_t(K) = K$, for all t ($\varphi_t(K) \subset K$, for all $t \geq 0$).

Theorem 4.3.6. *Suppose* (3.16) *holds on* $D \subset \mathbf{R}^n$. *Then any rectifiable* k*-surface* φ *such that* $\varphi(\overline{U})$ *is invariant with respect to* (1.1) *has* k*-dimensional Hausdorff measure* $\mathcal{H}^k \varphi(\overline{U}) = 0$.

With an additional assumption about the set D where (3.16) holds, we draw the conclusion that certain normal $(k-1)$-boundary in D cannot be invariant, generalizing Dulac's Criterion.

Theorem 4.3.7. *Suppose*

(*a*) (3.16) *holds on a bounded open positively invariant set* $D \subset \mathbf{R}^n$.

(*b*) ψ *is a simple closed* $(k-1)$*-surface such that* $\psi = \partial\varphi_0$ *for some rectifiable* k*-surface* φ_0 *in* D.

Then $\psi(\partial U)$ *is not invariant with respect to* (1.1).

Theorem 4.3.8. *Suppose*

(*a*) (3.16) *holds on an open set* $D \subset \mathbf{R}^n$.

(*b*) ψ *is a normal* $(k-1)$*-boundary such that* $\psi = \partial\varphi_0^n$ *for a sequence of rectifiable* k*-surfaces* $\varphi_0^n : U \to K$ *which satisfy* $\lim_{n\to\infty} \mathcal{S}\varphi_0^n = m$, *where* K *is a compact subset of* D *and* m *is the infimum of* $\mathcal{S}\varphi$ *over all* k*-surfaces* φ *in* D *with* $\varphi(\partial U) = \psi(\partial U)$.

Then $\psi(\partial U)$ *is not invariant with respect to* (1.1).

We prove both theorems with the aid of the following lemma which may be established using Stokes' Theorem as in the case $k = 2$ (See Proposition 4.2.2 in section §4.2).

Lemma 4.3.9. *Suppose that* $\varphi : \overline{U} \to \mathbf{R}^n$ *is a rectifiable* k*-surface in* $\mathbf{R}^n$ *such that* $\partial\varphi$ *is a normal* $(k-1)$*-boundary. Then there is a constant* $\delta > 0$ *which depends only on* $\varphi(\partial U)$ *such that the* k*-dimensional volume* $\mathcal{A}\varphi \geq \delta$.

Suppose that ψ is invariant. Then the conditions of Theorem 4.3.7 and Lemma 4.3.9 applied to (3.15), since $\varphi_t(\partial U) = \varphi_0(\partial U) = \psi(\partial U)$, imply $D_t^+ \mathcal{S}\varphi_t \leq -\eta < 0$ for all $t \geq 0$, where η is a constant. Therefore $\lim_{t\to\infty} \mathcal{S}\varphi_t = -\infty$, contradicting $\mathcal{S}\varphi_t > 0$. Thus no such invariant $(k-1)$-surface ψ can exist. Similarly, the conditions of Theorem 4.3.8 imply $D_t^+ \mathcal{S}\varphi_t^n \leq -\eta < 0, \quad 0 \leq t \leq \varepsilon$, for some $\varepsilon > 0$ and $n = 1, 2, \ldots$. so that $\mathcal{S}\varphi_\varepsilon^n \leq \mathcal{S}\varphi_0^n - \eta\varepsilon$ which implies $\limsup_{n\to\infty} \mathcal{S}\varphi_\varepsilon^n \leq m - \eta\varepsilon < m$, contradicting the definition of m, if $\varphi_\varepsilon^n(\partial U) = \varphi_0^n(\partial U) = \psi(\partial U)$.

§4.4. Orbital Stability of Periodic Solutions

Our main result in this section concerns the orbital stability of a periodic orbit of the system (1.1). We first recall the basic definitions (see [6]). Suppose (1.1) has a periodic solution $x = p(t)$ with least period $\omega > 0$ and orbit $\gamma = \{p(t) : 0 \leq t \leq \omega\}$. This orbit is *orbitally stable* if, for each $\epsilon > 0$, there exists a $\delta > 0$ such that any solution $x(t)$, for which the distance of $x(0)$ from γ is less δ, remains at a distance less than ϵ from γ for all $t \geq 0$. It is *asymptotically orbitally stable* if the distance of $x(t)$ from γ also tends to zero as $t \to \infty$. This orbit γ is *asymptotically orbitally stable with asymptotic phase* if it is asymptotically orbitally stable and there is a $b > 0$ such that, any solution $x(t)$, for which the distance of $x(0)$ from γ is less than b, satisfies $|x(t) - p(t-\tau)| \to 0$ as $t \to \infty$ for some τ which may depend on $x(0)$.

When $n = 2$, the following orbital stability criterion for periodic solutions is due to Poincaré.

Theorem 4.4.1 (Poincaré's Stability Criterion). *When* $n = 2$, *a periodic orbit* $\gamma = \{p(t) : 0 \leq t \leq \omega\}$ *of* (1.1) *is asymptotically orbitally stable with*

asymptotic phase if

$$\int_0^\omega \operatorname{div} f(p(t))\, dt < 0. \tag{4.1}$$

Using the theory of compound equations as is developed in Appendix B, the following result, which is a generalization of Poincaré's stability criterion to higher dimensional systems, is proved in [8].

Theorem 4.4.2. *A sufficient condition for a periodic orbit* $\gamma = \{p(t) : 0 \le t \le \omega\}$ *of* (1.1) *to be asymptotically orbitally stable with asymptotic phase is that the linear system*

$$z'(t) = \left(\frac{\partial f}{\partial x}^{[2]}(p(t))\right) z(t) \tag{4.2}$$

be asymptotically stable.

Concrete sufficient conditions for orbital stability of periodic orbits can be derived using the Theorem A.4.2 of Appendix A. The following is a typical example.

Corollary 4.4.3. *Suppose that, for some Lozinskiĭ measure* μ,

$$\int_0^\omega \mu\left(\frac{\partial f}{\partial x}^{[2]}(p(t))\right) dt < 0. \tag{4.3}$$

Then γ *is orbitally asymptotically stable with asymptotic phase.*

Remark. When $n = 2$, we know

$$\mu\left(\frac{\partial f}{\partial x}^{[2]}(p(t))\right) = \operatorname{div} f(p(t))$$

for all Lozinskiĭ measures. Therefore (4.3) is the Poincaré's criterion in this case.

Proof of Theorem 4.4.2. Let $x = p(t)$ be a nontrivial ω-periodic solution of (1.1). Then the variational equation

$$y'(t) = \frac{\partial f}{\partial x}(p(t))\, y(t) \tag{4.4}$$

with respect to $p(t)$ is a linear system with a ω-periodic coefficient matrix. By Floquet's theorem (see [7] p. 47), a fundamental matrix $Y(t)$ of (4.4) may be written in the form

$$Y(t) = P(t)\exp(Lt), \tag{4.5}$$

where the $n \times n$ matrices $P(t)$, L are ω-periodic and constant, respectively. The stability character of (4.4) is, therefore, determined by the eigenvalues of L which are called the *characteristic exponents.* Since $y = p'(t)$ is a nontrivial ω-periodic solution of (4.4), it follows that one of the characteristic exponents is equal to zero $(\operatorname{mod} 2\pi i/\omega)$. A fundamental result in stability theory is that γ is asymptotically orbitally stable with asymptotic phase if the remaining $(n-1)$ characteristic exponents have negative real part. Now the second compound equation (4.2) has fundamental matrix

$$Y^{(2)}(t) = P^{(t)}\exp\left(L^{[2]}t\right)$$

by Theorem B.3.1 and Proposition B.2.10 in Appendix B. The characteristic exponents of (4.2) are thus the eigenvalues of $L^{[2]}$ which are sums of pairs of eigenvalues of L (see Proposition B.2.6 of Appendix B). Since L has at least one eigenvalue zero, it follows that all the remaining $(n-1)$ eigenvalues of L are also eigenvalues of $L^{[2]}$. These eigenvalues must, therefore, all have negative real part since $Y^{(2)}(t) \to 0$, $t \to \infty$. Hence, γ is asymptotically orbitally stable. □

Remark. There are several different proofs in the literature for the fundamental result that all the remaining $(n-1)$ characteristic exponents of (4.4) have negative real part imply γ is asymptotically orbitally stable with asymptotic phase; for example, Coppel's proof involves exponential dichotomy, contraction mapping principle and reduction of order [3]; reduction of the dimension is also made possible in Hale's proof by construction of radial and angular coordinates around the periodic orbit [6]; Hartman [7] uses his linearization and invariant manifold techniques in his proof.

An application of this result will be seen in Chapter VII, where it is used together with Poincar'e–Bendixson property to show the nonexistence of nonconstant periodic solutions for some concrete models from Mathematical Biology.

§4.5. Periodic Solution of Periodic Systems

In this section we consider a ω-periodic system

$$x' = f(t,x) \tag{5.1}$$

where $(t,x) \mapsto f(t,x)$ is a function defined for $(t,x) \in \mathbf{R} \times D$, for some open set $D \subset \mathbf{R}^n$, such that solutions to (5.1) exist and are uniquely determined by their initial conditions. We assume that f is ω-periodic in t, namely

$$f(t+\omega, x) = f(t,x) \qquad \text{for all} \quad (t,x) \in \mathbf{R} \times D. \tag{5.2}$$

Let $\mathcal{P}$ be the Poincaré map associated with (5.1). We assume that $\mathcal{P}$ is dissipative with a bounded absorbing set $D_0 \subset D$. Then

$$\mathbf{A} = \cap_{n=1}^{\infty} \mathcal{P}^n(\bar{D}_0) \tag{5.3}$$

is the global attractor of $\mathcal{P}$ in D.

We have seen from Chapter II that if D is simply connected then $\dim_H \mathbf{A} < 2$ implies no simple closed rectifiable curve in $\mathbf{A}$ can be invariant under $\mathcal{P}$. As the following theorem demonstrates, this has strong implication for the asymptotic behaviour of solutions to (5.1).

Theorem 4.5.1. *Suppose D is simply connected. If $\dim_H \mathbf{A} < 2$ then any periodic solution of (5.1) has period commensurate with ω.*

In order to prove this theorem, we need the following Lemma.

Lemma 4.5.2. *Suppose α and β are two incommensurate real numbers. Then the set*

$$S = \{ n\,\alpha \,(\operatorname{mod} \beta) \;:\; n \in \mathbf{Z} \} \tag{5.4}$$

is dense in the interval $[0, \beta]$.

Proof. We will prove the lemma for the case when $\beta = 1$ and $\alpha > 0$ is irrational. The general case can be proved by considering $\alpha' = \alpha/\beta$. Now, since α is irrational, there exists a $\delta > 0$ such that, for any $0 \le \gamma < 1$, there exists a sequence of integer pairs (p_k, q_k) with $q_k \to 0$ as $k \to \infty$ such that

$$|\alpha - \frac{p_k}{q_k} - \frac{\gamma}{q_k}| < \frac{\delta}{q_k^2}$$

Therefore $q_k\alpha = p_k + \gamma + \delta q_k^{-1} = \gamma + \delta q_k^{-1} \pmod 1$. Thus $q_k\alpha \to \gamma$ as $k \to \infty$. As a result, S is dense in [0,1]. □

Proof of Theorem 4.5.1. Suppose that (5.1) has a nonconstant periodic solution of least period ω' incommensurate with ω. Denote this periodic solution by $x = x(t, x_0)$ such that $x(0) = x_0$ and its trajectory in $\mathbf{R}^n$ by

$$\Gamma = \{ x(t, x_0) \ : \ 0 \le t < \omega' \}.$$

Now, as a subset of $\mathbf{R}^n$, Γ is a simple closed smooth curve. We want to show that Γ is invariant under $\mathcal{P}$. For this we consider the set

$$B = \{ \mathcal{P}^n(x_0) \ : \ n \in \mathbf{Z} \}.$$

B is invariant under $\mathcal{P}$ since it is a complete orbit through x_0. Moreover, B is an infinite set, for $\mathcal{P}^n x_0 = x(n\omega, x_0)$ and ω' is incommensurate with ω. Next we know from the ω'-periodicity of $x(t, x_0)$, $\mathcal{P}^n x_0 = x(n\omega, x_0) = x([n\omega], x_0)$ where $[t] = t \pmod{\omega'}$, for each $t \in \mathbf{R}$. From Lemma 4.5.2, we know that the set $\{[n\omega] \ : \ n \in \mathbf{Z}\}$ is dense in $[0, \omega']$. Thus $B = \{x([n\omega], x_0) \ : \ n \in \mathbf{Z}\}$ is dense in Γ. This implies that the simple closed smooth curve Γ is invariant under $\mathcal{P}$. But this is impossible under the assumption $\dim_H \mathbf{A} < 2$. Therefore the theorem is proved. □

When f is C^1, the following result follows from an upper estimate for the Hausdorff dimension of A , Theorem 3.2.1 when $s = 0$, $k = 2$, in Chapter III.

Theorem 4.5.3. *Assume that* D_0 *is simply connected. If*

$$\int_0^{\omega} \mu \left(\frac{\partial f^{[2]}}{\partial x} (t, x) \right) dt < 0 \tag{5.5}$$

for all $x \in A$. *Then any periodic solution of* (5.1) *has a period commensurate with* ω.

Proof. The condition (5.5) implies $\dim_H \mathbf{A} < 2$, by Theorem 3.2.1 of Chapter III. □

Conjecture. *Under the assumptions of Theorem* 4.5.1 *or Theorem* 4.5.3, *any almost-periodic solution of* (5.1) *is periodic with a period commensurate with* ω.

§4.6. An Example

In this section, we consider the Lorenz model given in the section §1.4 of Chapter I and establish regions which contains no invariant closed curves and, in particular, no periodic trajectories. The system is

$$\begin{aligned} x' &= -\sigma x + \sigma y \\ y' &= r x - y - x z \\ z' &= -b z + x y \end{aligned} \tag{6.1}$$

where σ, r, b are three positive numbers.

This dissipative system has been the object of intensive numerical investigation which indicates that it has a strange attractor for a large range of values of the parameters and that it has many periodic orbits (see [11], page 21). Any set not containing periodic trajectories could therefore not completely contain the attractor.

We have seen in the section §3.4 of Chapter III that the Jacobian matrix J of

the right-hand side of (6.1) and its additive compounds are

$$J = \begin{bmatrix} -\sigma & \sigma & 0 \\ r-z & -1 & -x \\ y & x & -b \end{bmatrix} \tag{6.2}$$

$$J^{[2]} = \begin{bmatrix} -\sigma-1 & -x & 0 \\ x & -b-\sigma & \sigma \\ -y & r-z & -b-1 \end{bmatrix} \tag{6.3}$$

$$J^{[3]} = \operatorname{tr} J = -\sigma - b - 1. \tag{6.4}$$

First of all, (6.4) gives rise to $\mu(J^{[3]}) = \operatorname{tr} J < 0$. Hence Theorem 4.3.8 of §4.3 of this chapter implies that (6.1) has no invariant normal 2-boundaries.

Next we will try to identify regions which contain no invariant simple closed curves.

If we choose $S(x,y) = |y|$, where $|y| = \sup\{\sqrt{y_1^2 + y_2^2},\ |y_3|\}$, we find that

$$\mu(\frac{\partial f}{\partial x}^{[2]}) \le \sup\{-1, -b, -b-1+|y|+|z-r|\}$$

so that $\mu(\frac{\partial f}{\partial x}^{[2]}) < 0$ if

$$|y| + |z-r| < b+1, \tag{6.5}$$

which determines a cylinder D_0 parallel to the x-axis. In this case the functional $\mathcal{S}$ defined by (2.3) is

$$\mathcal{S}\varphi = \int_U \sup\left\{\left[\frac{\partial(\varphi_1,\varphi_2)}{\partial(u_1,u_2)}^2 + \frac{\partial(\varphi_1,\varphi_3)}{\partial(u_1,u_2)}^2\right]^{\frac{1}{2}},\ \left|\frac{\partial(\varphi_2,\varphi_3)}{\partial(u_1,u_2)}\right|\right\}. \tag{6.6}$$

The cylinder D_0 has the minimum property with respect to $\mathcal{S}$. To see this, suppose ψ is a simple closed rectifiable curve in the half-space $D : y + (z-r) \le c$. Let $\varphi \in \Sigma(\psi, \mathbf{R}^3)$ and suppose that $\varphi(\overline{U})$ crosses the plane $\Pi : y + (z-r) = c$, specifically $\varphi_2(u) + (\varphi_3(u) - r) > c$ if $u \in U_0 \subset \overline{U}$. Now consider the surface

$\widetilde{\varphi} \in \Sigma(\psi, D)$ obtained by modifying φ so that this portion is reflected in Π :

$$\widetilde{\varphi}(u) = \varphi(u), \qquad \text{if } u \in U \setminus U_0$$

$$\widetilde{\varphi}_1(u) = \varphi_1(u), \quad \widetilde{\varphi}_2(u) = c + r - \varphi_3(u), \quad \widetilde{\varphi}_3(u) = c - \varphi_2(u), \qquad \text{if } u \in U_0.$$

Then $\widetilde{\varphi} \in \Sigma(\psi, D)$, $\widetilde{\varphi}(\overline{U})$ does not cross Π and, from (6.6), $\mathcal{S}\widetilde{\varphi} = \mathcal{S}\varphi$. A similar argument may be applied to planes $\pm y \pm (z - r) = c$ and $x = k$ to deduce that, if ψ is a simple closed rectifiable curve in D_0, then there is a compact box in D_0 which contains the curve and a minimizing sequence for $\mathcal{S}$ in $\Sigma(\psi, D_0)$.

We conclude from Theorem 2.4 that the Lorenz system (6.1) has no invariant rectifiable closed curve in the cylinder defined by (6.5).

This statement may be improved by a more judicious choice of S. Let $S(x, y) = |Ay|$ where $A = \operatorname{diag}(1, 1, \alpha)$ and α is a positive constant; in fact

$$S(x, y) = \sup\{\sqrt{y_1^2 + y_2^2},\ \alpha|y_3|\},$$

another norm. Now

$$\mu(A\frac{\partial f}{\partial x}^{[2]} A^{-1}) \leq \sup\{-1 + (\frac{1}{\alpha} - 1)\sigma,\ -b + (\frac{1}{\alpha} - 1)\sigma,\ -b - 1 + \alpha(|y| + |z - r|)\}.$$

By optimizing the choice of the constant α we find $\mu(A\frac{\partial f}{\partial x}^{[2]} A^{-1}) < 0$ if

$$|y| + |z - r| < (b + 1)\inf\left\{\frac{\sigma + 1}{\sigma}, \frac{\sigma + b}{\sigma}\right\}, \tag{6.7}$$

a larger cylinder than that defined by (6.5).

A similar argument to the one given previously now shows: *The Lorenz system* (6.1) *has no invariant rectifiable closed curve in the cylinder defined by* (6.7).

In conclusion we note that the choice $S(x, y) = |y| = \sup\{\sqrt{y_1^2 + y_2^2}, |y_3|\}$ leading to the "surface area" $\mathcal{S}$ in (6.6) gives stronger results than the more usual norms $|y| = \sup\{|y_1|, |y_2|, |y_3|\}$, $|y| = |y_1| + |y_2| + |y_3|$ or $|y| = (y_1^2 + y_2^2 + y_3^2)^{1/2}$. The first two, by easy computations, lead to the conditions (i), (ii) of Theorem 4.2.3 which, for the Lorenz system, hold on smaller sets than that specified by (6.5). The third norm requires an estimation of the region where the expression $\lambda_1 + \lambda_2$ of

Theorem 4.2.3 (iii) is negative and this may be implemented using the approach of Smith [9] but also leads to a smaller set than (6.7).

§4.7. Bibliography for Chapter IV

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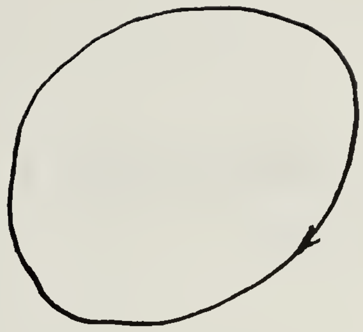
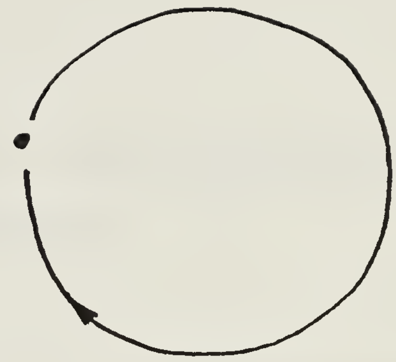
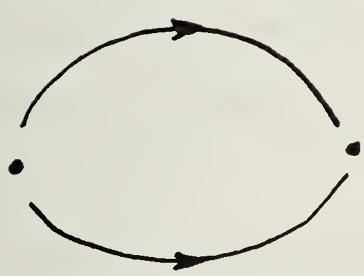


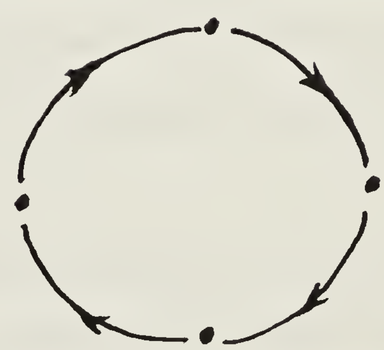
(1) A perioic orbit.



(2) A homoclinic orbit.



(3) A pair of heteroclinic orbits.



(4) A heteroclinic cycle.

Figure 4.1.1.

CHAPTER V

A GEOMETRIC APPROACH TO THE PROBLEM OF GLOBAL STABILITY

In this chapter, we develop a new approach to the problem of global stability for autonomous systems of differential equations. The approach is based on our Dulac criteria discussed in Chapter IV and the C^1 Closing Lemma of Pugh [14] (see also [13], [15]). It differs from the traditional method of constructing Lyapunov functions in that it provides concrete and flexible conditions which can be computed directly from the equations.

We start in §5.1 by introducing the notion of robustness for a Bendixson Criterion under C^1 local perturbations of the vector field at every nonwandering point. Then the Closing Lemma is used to show that if the autonomous system satisfies a Bendixson Criterion which has this robustness, then every nonwandering point is an equilibrium. This implies, for example, that every alpha and omega limit point is an equilibrium if such a Bendixson Criterion is satisfied. In particular, if the system is dissipative and has a unique equilibrium $\bar{x}$ which is also locally stable, then this assumption will imply $\bar{x}$ attracts every point, and thus is globally asymptotically stable.

In §5.2, we show that the Dulac criteria derived in Chapter IV all have this required robustness. The Centre Manifold Theorem is then used to show that these criteria imply each bounded trajectory converges to an equilibrium. We also prove that they place severe restrictions on the structure of compact invariant sets.

The problem of global stability is taken in §5.3, where we prove that, if $\bar{x}$ is the unique equilibrium, then $\bar{x}$ is locally stable if any of our Dulac criteria holds. Therefore these criteria will also imply the global stability of $\bar{x}$.

Then in §5.4, we show how similar conclusions can be drawn when the Dulac inequality is not strict. An example is given to illustrate the general theory.

We will see in Chapter VII that this new approach is applied to resolve an open

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problem in Mathematical Biology.

§5.1. The Closing Lemma and Wandering Point Conditions

Let the mapping $x \mapsto f(x)$ from an open subset D of $\mathbf{R}^n$ to $\mathbf{R}^n$ be such that each solution $x(t)$ to the differential equation

$$x' = f(x) \tag{1.1}$$

is uniquely determined by its initial value $x(0) = x_0$ and denote this solution $x(t, x_0)$.

A point $x_0 \in D$ is *wandering* for (1.1) if there exists a neighbourhood U of x_0 and $T > 0$ such that $U \cap x(t, U)$ is empty for all $t > T$. Thus, for example, any equilibrium, alpha limit point or omega limit point is nonwandering.

We first of all formulate the local version of the C^1-Closing Lemma of Pugh which plays an essential role in the development of this chapter. Let $|\cdot|$ denote a vector norm on $\mathbf{R}^n$ and the operator norm it induces for linear mappings from $\mathbf{R}^n$ to $\mathbf{R}^n$. The distance between two functions $f, g \in C^1(D \to \mathbf{R}^n)$ such that $f - g$ has compact support is

$$|f - g| = \sup\{|f(x) - g(x)| + |Df(x) - Dg(x)| : x \in D\}. \tag{1.2}$$

A function $g \in C^1(D \to \mathbf{R}^n)$ is called a C^1 *local ϵ-perturbation* of f at $x_0 \in D$ if there exists an open neighbourhood U of x_0 in D such that $\operatorname{supp}(f - g) \subset U$ and $|f - g| < \epsilon$. For such a g we consider the corresponding differential equation

$$x' = g(x). \tag{1.3}$$

Lemma 5.1.1. *Let $f \in C^1(D \to \mathbf{R}^n)$. Suppose that x_0 is a nonwandering point for (1.1) and that $f(x_0) \neq 0$. Then, for each neighbourhood U of x_0 and $\epsilon > 0$, there exists a C^1 local ϵ-perturbation g of f at x_0 such that*

(1) $\operatorname{supp}(f - g) \subset U$, *and*

(2) *the system* (1.3) *has a nonconstant periodic solution whose trajectory intersects* U.

A *Bendixson Criterion* for (1.1) is a condition satisfied by f which precludes the existence of nonconstant periodic solutions to (1.1). A Bendixson Criterion is said to be *robust under* C^1 *local perturbations* of f at x_0, if for each sufficiently small $\epsilon > 0$ and neighbourhood U of x_0, it is also satisfied by all C^1 local ϵ-perturbations g such that $\operatorname{supp}(f-g) \subset U$.

Suppose now that f satisfies a Bendixson Criterion which is robust under C^1 local perturbations of f at all nonwandering points of (1.1) which are not equilibria. Then, for each C^1 local ϵ-perturbation g of f at such a nonwandering point and when ϵ is sufficiently small, (1.3) can not have any nonconstant periodic solutions. Therefore Lemma 5.1.1 implies that every nonequilibrium point of (1.1) must be wandering. We thus have the following general wandering point theorem.

Theorem 5.1.2. *Suppose a Bendixson Criterion for* (1.1) *is robust under* C^1 *local perturbations of* f *at all nonequilibrium nonwandering points to* (1.1). *Then every nonequilibrium point of* (1.1) *is wandering.*

As a special case, the following result follows directly from Theorem 5.1.2.

Corollary 5.1.3. *Suppose that* (1.1) *satisfies a Bendixson Criterion that is robust under* C^1 *perturbations of* f *at all points of* D. *Then every nonwandering point of* (1.1) *is an equilibrium.*

Suppose $D = \mathbf{R}^n$ and all solutions to (1.1) are bounded. Then for each $x_0 \in \mathbf{R}^n$, $\omega(x_0)$ is nonempty and compact. Now assume that (1.1) has a unique equilibrium $\bar{x}$ in $\mathbf{R}^n$. Then the conditions of Theorem 5.1.2 imply that $\omega(x_0) = \bar{x}$ for all $x_0 \in \mathbf{R}^n$. If moreover $\bar{x}$ is a stable equilibrium of (1.1), then it is globally asymptotically stable. As a matter of fact $\{\bar{x}\}$ is the global attractor of (1.1) in $\mathbf{R}^n$. Therefore $\bar{x}$ is a globally asymptotically stable equilibrium. We thus have the following general result on global stability.

Theorem 5.1.4. *Assume that*

(1) $D = \mathbf{R}^n$ *and all solutions to* (1.1) *are bounded,*

(2) $\bar{x} \in \mathbf{R}^n$ *is the unique equilibrium of* (1.1) *in* $\mathbf{R}^n$,

(3) $\bar{x}$ *is stable,*

(4) (1.1) *satisfies a Bendixson Criterion that is robust under* C^1 *local perturbations of* f *at each nonwandering point for* (1.1) *which is not an equilibrium.*

Then $\{\bar{x}\}$ *is the global attractor of* (1.1) *in* $\mathbf{R}^n$. *In particular,* $\bar{x}$ *is globally asymptotically stable in* $\mathbf{R}^n$.

When $D \subset \mathbf{R}^n$ is an open subset, results like Theorem 5.1.4 also holds if D contains an absorbing set. Recall that a subset $D_0 \subset \bar{D}_0 \subset D$ is *absorbing* with respect to (1.1) if solutions exist for all $t \geq 0$ and each compact subset D_1 of D satisfies $x(t, D_1) \subset D_0$ for all sufficiently large t. System (1.1) is *dissipative* if it has a bounded absorbing set $D_0 \subset D$. Since the trajectory of every solution eventually enters D_0 and stays there, it does not approach the boundary of D. The conditions of Theorem 5.1.2 imply that its omega limit set is a singleton $\{\bar{x}\}$ where $\bar{x}$ is the unique equilibrium. Therefore we have the following local version of Theorem 5.1.4.

Theorem 5.1.5. *Suppose that*

(1) *there exists a bounded absorbing set* $D_0 \subset D$,

(2) $\bar{x} \in D_0$ *is the unique equilibrium of* (1.1) *in* D,

(3) $\bar{x}$ *is stable,*

(4) (1.1) *satisfies a Bendixson Criterion that is robust under* C^1 *local perturbations of* f *at all nonequilibrium nonwandering points for* (1.1).

Then $\{\bar{x}\}$ *is the global attractor of* (1.1) *in* D. *In particular,* $\bar{x}$ *is globally asymptotically stable in* D.

Remarks.

(i). The condition (1) of Theorem 5.1.5 can be weakened so that the bounded subset D_0 only attracts all points of D. Under this weaker condition $\{\bar{x}\}$ also attracts all points of D. This and the stability of $\bar{x}$ imply that $\{\bar{x}\}$ attracts every compact subset of D, and thus is the global attractor in D.

(ii). We want to note that when $\bar{x}$ is hyperbolic, its stability can be easily determined by linearizing (1.1) at $\bar{x}$. Therefore condition (3) is easy to check. Theorem 5.1.4 and Theorem 5.1.5 thus provides a new and practical approach to the problem of global stability.

(iii). In many cases a Bendixson Criterion would imply that the unique equilibrium $\bar{x}$ is locally stable. We shall see that all the criteria we will discuss in the next section have this property, thus the local stability assumption (3) on $\bar{x}$ can be dropped in these cases. However, there are conditions which preclude nonconstant periodic solutions but may not imply the local stability of $\bar{x}$.

§5.2. Autonomous Convergence Theorems

In this section, we examine some Bendixson criteria, as well as criteria of Dulac type discussed in Chapter IV. We will show that they are robust under C^1 local perturbations of f and thus imply that all nonwandering points of (1.1) are equilibria. Moreover, we will show that these conditions greatly restrict the structure of limit sets of (1.1).

Let B denote the euclidean unit ball in $\mathbf{R}^2$ and $\overline{B}$, ∂B its closure and boundary respectively. Recall that a function $\varphi \in \mathrm{Lip}(\overline{B} \to D)$ is considered a *simply connected rectifiable* 2-*surface* in D or, briefly, *a surface* in D; a function $\psi \in \mathrm{Lip}(\partial B \to D)$ is a *closed rectifiable curve* in D, will be called *simple* if it is one-to-one and we write $\psi = \partial\varphi$ if $\varphi(\partial B) = \psi(\partial B)$.

Let $x \mapsto A(x)$ be a nonsingular $\binom{n}{2} \times \binom{n}{2}$ matrix-valued function which is C^1 on D and let $|\cdot|$ be a norm on $\mathbf{R}^{\binom{n}{2}}$. We consider a functional $\mathcal{S}$ on the

surfaces in D defined by

$$\mathcal{S}\varphi = \int_B |A(\varphi)\, \frac{\partial \varphi}{\partial u_1} \wedge \frac{\partial \varphi}{\partial u_2}| \tag{2.1}$$

if $u = (u_1, u_2)$ and $u \mapsto \varphi(u)$ is lipschitzian on $\overline{B}$. If $|y|^2 = y^* y$ and $A(x) = I$, then $\mathcal{S}\varphi$ is the usual surface area of $\varphi(\overline{B})$ counting multiplicities. The set D has the *minimum property* with respect to $\mathcal{S}$ if, for each simple closed rectifiable curve ψ in D, there is a sequence of surfaces φ^k in D which is a minimizing sequence for $\mathcal{S}\varphi$ with respect to all surfaces φ in D with $\psi = \partial\varphi$ and such that $\cup_k \varphi^k(\overline{B})$ has compact closure in D. If $n = 2$, then any simply connected open set D has the minimum property with respect to any $\mathcal{S}$, since $\mathcal{S}\varphi = \int_{\varphi(B)} |A|$. When $n \geq 2$, for example, if $\mathcal{S}$ is the usual surface area, then any convex open set D has the minimum property. The set $D = \mathbf{R}^n$ has the minimum property with respect to $\mathcal{S}$ if $A = I$ and $|\cdot|$ is any absolute norm. In fact, if φ^k is a sequence with $\psi = \partial\varphi^k$ which minimizes $\mathcal{S}$ in this case, then for any interval $\mathcal{I} \subset \mathbf{R}^n$ with $\psi(\partial B) \subset \mathcal{I}$ we can obtain a sequence $\widetilde{\varphi}^k$ with $\mathcal{S}\widetilde{\varphi}^k = \mathcal{S}\varphi^k$ and $\widetilde{\varphi}^k(\overline{B}) \subset \mathcal{I}$, by reflection in the sides of the interval $\mathcal{I}$.

For a simply connected open set D which has the minimum property with respect to $\mathcal{S}$, we will assume that the generalized Dulac condition is satisfied:

$$\mu(A_f A^{-1} + A\, \frac{\partial f}{\partial x}^{[2]} A^{-1}) < 0 \quad \text{in} \quad D. \tag{2.2}$$

Here μ is the Lozinskiĭ measure corresponding to the norm $|\cdot|$ on $\mathbf{R}^{\binom{n}{2}}$ considered in (2.1), $A_f = (DA)(f)$ or, equivalently, A_f is the matrix obtained by replacing each entry a_{ij} in A by its directional derivative in the direction f, $\frac{\partial a_{ij}}{\partial x}^* f$, and $\frac{\partial f}{\partial x}^{[2]}$ is a $\binom{n}{2} \times \binom{n}{2}$ matrix, the second additive compound of the Jacobian matrix $\frac{\partial f}{\partial x}$. The condition (2.2) is equivalent to assuming that $V(x,y) = |A(x)y|$ is a Lyapunov function whose derivative with respect to the $n + \binom{n}{2}$ dimensional system

$$\frac{dx}{dt} = f(x), \qquad \frac{dy}{dt} = \frac{\partial f}{\partial x}^{[2]}(x)y \tag{2.3}$$

is negative definite.

When D does not necessarily have the minimum property with respect to $\mathcal{S}$, it will be assumed that there is a set $D_0 \subset D$ which is absorbing with respect to (1.1) such that

$$\mu(A_f A^{-1} + A \frac{\partial f^{[2]}}{\partial x} A^{-1}) \leq -\delta < 0 \quad \text{in} \quad D_0. \tag{2.4}$$

Proposition 5.2.1. *If* (2.2) *is satisfied in* D, *then the dimension of the stable manifold of any equilibrium is at least* $(n-1)$. *If an equilibrium is not isolated, then its stable manifold has dimension* $(n-1)$ *and it has a local centre manifold of dimension* 1 *which contains all nearby equilibria.*

Proof. For the definitions of centre manifold and stable or unstable manifold, see [10]. If x_1 is an equilibrium, then

$$\mu(A \frac{\partial f^{[2]}}{\partial x} A^{-1}) = \mu(A_f A^{-1} + A \frac{\partial f^{[2]}}{\partial x} A^{-1}) < 0 \tag{2.5}$$

at x_1, since $f(x_1) = 0$ implies $A_{f(x_1)}(x_1) = 0$. If $\nu_i(x_1)$ are the eigenvalues of $\frac{\partial f}{\partial x}(x_1)$ with $\text{Re}\,\nu_1(x_1) \geq \text{Re}\,\nu_2(x_1) \geq \cdots \geq \text{Re}\,\nu_n(x_1)$, then $\nu_i(x_1)+\nu_j(x_1)$, $i \neq j$, are the eigenvalues of $\frac{\partial f}{\partial x}^{[2]}(x_1)$ and hence of $A \frac{\partial f}{\partial x}^{[2]} A^{-1}(x_1)$. Thus (2.5) implies $\text{Re}\,[\nu_i(x_1) + \nu_j(x_1)] \leq \mu[A \frac{\partial f}{\partial x}^{[2]} A^{-1}(x_1)] < 0$ (see Proposition A.2.1, Appendix A); therefore $0 > \text{Re}\,\nu_2(x_1) \geq \cdots \geq \text{Re}\,\nu_n(x_1)$ and only $\nu_1(x_1)$ can possibly have nonnegative real part; the stable manifold has dimension at least $n-1$. If the equilibrium x_1 is not isolated, $\frac{\partial f}{\partial x}(x_1)$ is a singular matrix, $0 = \nu_1(x_1)$ so the stable manifold has dimension $(n-1)$ and there is a 1-dimensional centre manifold. Since all positive semitrajectories originating near x_1 are asymptotic to a trajectory in the centre manifold, all equilibria near x_1 are in the centre manifold.

Theorem 5.2.2. *Suppose* D *is simply connected,* f *is of class* C^1 *on* D *and there exists a* $\binom{n}{2} \times \binom{n}{2}$ *matrix-valued function* A *which is also* C^1 *on* D *and such that either* (a) *or* (b) *is satisfied:*

(a) D *has the minimum property with respect to* $\mathcal{S}$ *and* (2.2) *is satisfied.*

(*b*) D_0 *is an absorbing subset of* D *with respect to* (1.1) *and* (2.4) *is satisfied.*

Then:

(*c*) *Every nonwandering point of* D *is an equilibrium.*

(*d*) *Every nonempty alpha or omega limit set in* D *is a single equilibrium.*

(*e*) *Any equilibrium in* D *is the alpha limit set of at most two distinct nonequilibrium trajectories.*

Proof. If (a) or (b) is satisfied by f, then a similar condition is also satisfied by all C^1-perturbations g of f considered in Lemma 5.1.1. From the Criterion I and Criterion II in §4.2.2 of Chapter IV, no such perturbations can have a nonconstant periodic solution to (1.3). Therefore every nonwandering point is an equilibrium for (1.1).

To prove the assertion that each nonempty alpha or omega limit set is a single equilibrium, first observe that since each limit point is nonwandering, it is an equilibrium. Let $x_1 \in \omega(x_0)$, the omega limit set of x_0; if x_1 is an isolated equilibrium, then $\{x_1\} = \omega(x_0)$ since $\omega(x_0)$ is locally a continuum and, if it has more than one connected component, each component must connect to ∂D. If $x_1 \in \omega(x_0)$ is not an isolated equilibrium, then Proposition 5.2.1 implies there is a 1-dimensional centre manifold containing all nearby equilibria associated with x_1. Every trajectory which intersects a neighbourhood U of x_1 is asymptotic to a trajectory in the centre manifold. Thus $\lim_{t\to\infty} x(t, x_0) = x_1$ so that $\omega(x_0) = \{x_1\}$ in this case also. The proof that a nonempty alpha limit set is an equilibrium x_2 is the same. Moreover, since the stable manifold of x_2 has dimension $(n-1)$, x_2 has either a 1-dimensional centre manifold or a 1-dimensional unstable manifold. Since all trajectories near the centre manifold are asymptotic to a trajectory in that manifold ([12] P. 48) and the stable manifold is asymptotic to x_2, at most two nonequilibrium trajectories can share x_2 as their alpha limit. The uniqueness of the unstable manifold implies the same conclusion in the other case.

The equilibrium $\omega(x_0)$ need not be isolated as is seen from the example

$$\frac{dx_1}{dt} = -x_1, \quad \frac{dx_2}{dt} = 0.$$

Here $\frac{\partial f}{\partial x}^{[2]} = \operatorname{div} f = -1 < 0$ so that (2.2) is satisfied with $A = 1$. $D = \mathbf{R}^2$ has the minimum property with respect to $\mathcal{S}$, which is area in the plane in this case, and the conditions of Theorem 5.2.2 are all satisfied. Each solution satisfies $\lim_{t\to\infty}\big(x_1(t), x_2(t)\big) = \lim_{t\to\infty}\big(x_1(0)e^{-t}, x_2(0)\big) = \big(0, x_2(0)\big)$ a nonisolated equilibrium. □

When $A = I$, (2.2) becomes $\mu\big(\frac{\partial f}{\partial x}^{[2]}\big) < 0$. We thus have the following corollary.

Corollary 5.2.3. *The conclusion of Theorem* 5.2.2 *holds if* $D = \mathbf{R}^n$ *and the generalized Bendixson Criterion* $\mu\big(\frac{\partial f}{\partial x}^{[2]}\big) < 0$ *is satisfied in* $\mathbf{R}^n$, *where* μ *is the Lozinskiĭ measure corresponding to an absolute norm.*

Conditions (i), (ii), (iii) of Theorem 4.2.3 in Chapter IV give concrete examples of the condition $\mu\big(\frac{\partial f}{\partial x}^{[2]}\big) < 0$. The conditions (iv), (v), (vi) of this theorem are examples of $\mu\big(-\frac{\partial f}{\partial x}^{[2]}\big) < 0$ which, as we see below, has similar consequences.

Denote by $(1.1)_-$ the system (1.1) with f replaced by $-f$. The trajectories of $(1.1)_-$ are the same as those of (1.1) with the direction of the flow reversed. From this we deduce the following corollary.

Corollary 5.2.4. *If the system* (1.1) *-satisfies the conditions of Theorem 5.2.2 or Corollary 5.2.3, then the same conclusions may be drawn for* (1.1) *except that the statements about alpha and omega limit sets should be interchanged.*

Even in the case $n = 2$, this result gives a somewhat stronger conclusion than that usually drawn from Bendixson's criterion.

Corollary 5.2.5. *Suppose* $D \subset \mathbf{R}^2$ *is simply connected and*

$$\operatorname{div} f(x) < 0 \quad (> 0) \quad \textit{in} \quad D.$$

Then every nonwandering point with respect to (1.1) *is an equilibrium; every nonempty alpha or omega limit set is a single equilibrium; any equilibrium is the alpha (omega) limit set of at most two distinct nonequilibrium trajectories.*

A subset K of D is *positively* (*negatively*) *invariant* with respect to (1.1) if $x(t, K) \subset K$ for all $t \geq 0$ $(t \leq 0)$ and is *invariant* if $x(t, K) = K$ for all t. The alpha and omega limit sets of a trajectory are important examples of invariant sets and we have seen in the preceding discussion that these are very simple for systems which satisfy Dulac's Condition or its higher dimensional generalizations. In fact we will show that any compact set which is invariant in such a system is at most 1-dimensional.

Theorem 5.2.6. *Suppose that* f *satisfies the conditions of Theorem* 5.2.2 *and* $K \subset D$ *is a compact set which is invariant with respect to* (1.1). *Then its Hausdorff dimension satisfies*

$$\dim_H K \leq 1.$$

In particular, if K *is also connected, then* $\dim_H K = 0$ *or* 1 *depending on whether* K *contains one point or more than one point.*

Proof. Since K is compact, Theorem 5.2.2 implies that every trajectory in K is either an equilibrium or is asymptotic at both ends to an equilibrium and that every alpha limit is an isolated equilibrium. Let K_0 be the set of equilibria in K and K_0' its set of cluster points. If $x \in K_0'$, then Proposition 5.2.1 implies that there is a neighbourhood $U(x)$ of x such that all equilibria in $U(x)$ lie in a 1-dimensional C^1 local centre manifold $\sigma(x)$ at x. A finite set of neighbourhoods $U(x_i)$, $i = 1, \ldots, N$, covers the compact set K_0'. The set K is composed of complete trajectories of the following three types:

(i) trajectories in one of the 1-dimensional manifolds $\sigma(x_i)$, $i = 1, \ldots, N$,

(ii) the finite set of equilibria $K_0 \setminus \cup_i \sigma(x_i)$,

(ii) nonequilibrium trajectories whose alpha and omega limit sets are each single equilibria of type (i), (ii).

Any trajectory or any smooth arc has Hausdorff measure zero in dimension $s > 1$ (see Appendix C or [5]). From this it follows that the set of all trajectories of types (i), (ii) has s-dimensional measure zero. Moreover there are at most finitely many trajectories of type (iii). Otherwise there would exist a rectifiable simple closed curve composed of trajectories of types (ii) and (iii) together with a finite number of invariant arcs from $\sigma(x_i)$, $i = 1, \cdots, N$. It has been shown in Chapter IV that conditions of Theorem 5.2.2 preclude the existence of such invariant curves. We therefore conclude that the set of trajectories of type (iii) also has zero s-dimensional measure, if $s > 1$, and $\dim_H K \leq 1$ (see Appendix C). If K is also connected and contains more than one point, then the sum of the diameters of the sets in any open cover exceeds the distance between any pair of points in K which allows us to conclude $\dim_H K \geq 1$ and therefore $\dim_H K = 1$. □

Remarks.

(i). Smith, [16] Theorem 7, shows that, if $\lambda_1(x) \geq \lambda_2(x) \geq \cdots \geq \lambda_n(x)$ are the eigenvalues of $\frac{1}{2}[\frac{\partial f}{\partial x}^*(x) + \frac{\partial f}{\partial x}(x)]$, then each bounded semitrajectory of (1.1) converges to an equilibrium if $\lambda_1(x) + \lambda_2(x) < 0$ in $\mathbf{R}^n$. This also follows from Corollary 5.2.3 of this paper since $\mu(\frac{\partial f}{\partial x}^{[2]}) = \lambda_1 + \lambda_2$ if $|\cdot|$ is the euclidean norm on $\mathbf{R}^{\binom{n}{2}}$. Our result shows that the same conclusion can be drawn if the boundedness assumption is replaced by one of existence of an alpha or omega limit point of the semitrajectory. The domain $\mathbf{R}^n$ may be replaced by any convex open set D since such sets have the minimum property with respect to S which is the usual surface area in this case. The domain may also be any open set D which is simply connected and has an absorbing subset D_0 in which $\lambda_1(x) + \lambda_2(x) \leq -\delta < 0$ holds. Analogous results may be inferred from conditions of the form $\lambda_{n-1}(x) + \lambda_n(x) > 0$,

since $-\mu(-\frac{\partial f}{\partial x}^{[2]}) = \lambda_{n-1} + \lambda_n$. Smith's proof shows that his condition implies $\dim_H K < 2$ for any compact invariant K; in fact we see from Theorem 5.2.6 that his condition implies $\dim_H K \leq 1$.

(ii). An earlier result of Hartman and Olech [8] is somewhat related to observations of this paper. They show that if $x = 0$ is the only equilibrium of (1.1) and it is locally asymptotically stable, then it is globally asymptotically stable provided $\lambda_1(x) + \lambda_2(x) \leq 0$ in $\mathbf{R}^n$ and $\int_0^\infty p = \infty$ where $p(u) = \min\{|f(x)| : |x| = u\}$.

(iii). In a recent paper [2] on the celebrated Jacobian Conjecture, Drużowski and Tutaj assume that f is a polynomial map of $\mathbf{R}^n$ with $f(0) = 0$, $\frac{\partial f}{\partial x}(x)$ nonsingular for all x and all eigenvalues having negative real parts at equilibrium points. They prove that 0 is a globally asymptotically stable solution of (1.1) provided $\lambda_1 + \lambda_2 \leq 0$ for all $x \in \mathbf{R}^n$. It is interesting to speculate if their condition can be replaced by $\mu(\frac{\partial f}{\partial x}^{[2]}) < 0$ for an arbitrary Lozinskiĭ measure.

In the rest of this section, we assume that (1.1) is dissipative with a bounded absorbing set $D_0 \subset \bar{D}_0 \subset D$. Let $A(x)$ be the $\binom{n}{2} \times \binom{n}{2}$ matrix-valued function defined earlier, and let

$$B = A_f A^{-1} + A \frac{\partial f}{\partial x}^{[2]} A^{-1} \tag{2.5}$$

Let $|\cdot|$ be a vector norm on $\mathbf{R}^{\binom{n}{2}}$, and μ be the Lozinskiĭ measure corresponding to $|\cdot|$. For every solution $x(t, x_0)$ of (1.1) with $x_0 \in D_0$, we define the following quantities,

$$q(f, D_0, t) = \sup_{x_0 \in \overline{D}_0} \frac{1}{t} \int_0^t \mu(B(x(s, x_0)))\, ds \tag{2.6}$$

and

$$q(f, D_0) = \limsup_{t \to \infty} q(f, D_0, t). \tag{2.7}$$

These quantities are well-defined since D_0 is a bounded absorbing set.

A similar quantity $\bar{q}_k(K)$ is defined in (2.9) in Chapter III, where K is a compact invariant set and k is an integer. Here D_0 is a bounded absorbing set.

We know that $A = \omega(D_0) \subset D$ is the global attractor in D. Theorem 4.1.7 in Chapter IV establishes that $\bar{q}_2(A) < 0$ is a Bendixson Criterion. This implies $q(f, D_0) < 0$ is also a Bendixson Criterion. We will show in the following that it is robust under C^1 local perturbations of f at all nonequilibrium nonwandering points of (1.1). Once this is established, we are able to show that all the results obtained under the condition (2.2), namely $\mu(B) < 0$ pointwise in D, can be proved under the weaker condition $q(f, D_0) < 0$. Applications of this weaker condition will be seen in Chapter VII.

For a nonwandering point $x_0 \in D$ of (1.1) which is not an equilibrium, each sufficiently small neighbourhood U of x_0 initially leaves itself completely, along the solutions of (1.1), and will eventually come back to intersect itself. The smaller the U, the longer it takes for U to come back. The following quantities are then well defined.

$$\tau(U;\, x_0) =$$

$$\min\{\, t > 0 \;:\; x(t, U) \cap U \neq \emptyset, \text{ and } \exists\, t_1 < t \text{ such that } x(t_1, U) \cap U = \emptyset \,\} \tag{2.8}$$

and

$$\tau(x_0) = \sup\{\, \tau(U;\, x_0) \;:\; U \text{ is a sufficiently small neighbourhood of } x_0 \,\} \tag{2.9}$$

When x_0 is an equilibrium $\tau(x_0)$ is defined to be zero. We call $\tau(x_0)$ the *minimum return time* at the nonwandering point x_0. It follows from the continuous dependence on initial conditions that x_0 is an equilibrium if and only if $\tau(x_0) = 0$. In fact, we have the following result.

Lemma 5.2.7. *A solution $x(t, x_0)$ to (1.1) is periodic if and only if $\tau(x_0)$ is finite, in which case $\tau(x_0)$ is the minimum period.*

Proof. Suppose $x(t, x_0)$ is a periodic solution of least period T. If $T = 0$. then x_0 is an equilibrium and $\tau(x_0) = T = 0$. Suppose now $T > 0$. Then $\tau(U; x_0) \leq T$ for all sufficiently small neighbourhoods U of x_0 by the continuous dependence

of solutions on the initial conditions, and thus $\tau(x_0) \leq T$. If $\tau(x_0) < \bar{t} < T$, then there is a sequence of points $\{x_k\}$ such that $x_k \to x_0$, $k \to \infty$, and $x(\bar{t}, x_k) = x_k$, for each k. This leads to $x(\bar{t}, x_0) = x_0$, and thus contradicts the minimality of T.

Suppose now x_0 is nonwandering and $\tau(x_0) < \infty$. If x_0 is not an equilibrium, then $T = \tau(x_0) > 0$. There exists a sequence of points $\{x_n\}$ and a sequence of numbers $\{t_n\}$ such that $x_n \to x_0$ as $t \to \infty$, $T/2 \leq t_n \leq T$ for each n and $x(t_n, x_n) \to x_0$ as $n \to \infty$. Let $\{t_{n_k}\}$ be a convergent subsequence of $\{t_n\}$ and assume $t_{n_k} \to t_0$ as $n_k \to \infty$. Then $t_0 \geq T/2 > 0$. Moreover,

$$|x(t_0, x_0) - x_0| \leq |x(t_0, x_0) - x(t_{n_k}, x_{n_k})| + |x(t_{n_k}, x_{n_k}) - x_0|,$$

which implies $x(t_0, x_0) = x_0$, i.e. $x(t, x_0)$ is a periodic solution to (1.1). □

Lemma 5.2.8. *Suppose $\tau(x_0) = +\infty$. Then the condition $q(f, D_0) < 0$ is robust under C^1 local perturbations of f at x_0.*

Proof. Let $\delta = -q(f, D_0) > 0$. There exists a $T > 1$ such that

$$\int_0^t \mu\big(B(x(s, \bar{x}))\big)\, ds \leq -\frac{\delta t}{2} \tag{2.10}$$

for all $t > T$ and all $\bar{x} \in D_0$. The assumption $\tau(x_0) = +\infty$ implies that $f(x_0) \neq 0$ and $\tau(U; x_0) > T$ for all sufficiently small neighbourhoods U of x_0. Let Π be a transversal to the vector $f(x_0)$ at x_0 and E be a sufficiently small ball in Π centred at x_0. Consider the flow box

$$\Sigma = \{\, x(t, E) \; : \; -\alpha \leq t \leq \alpha \,\}$$

generated by the evolution of the ball $E \subset \Pi$ along the solutions of (1.1) for a small time interval $[-\alpha, \alpha]$ (see Figure 5.2.1). Let $\Gamma_+ = x(\alpha, E)$ and $\Gamma_- = x(-\alpha, E)$. By taking the ball $E \subset \Pi$ and $\alpha > 0$ sufficiently small, we can ensure that all solutions of (1.1) starting in Σ leave Σ and that $\tau(\Sigma; x_0) > T$. As a consequence, each solution starting at Γ_+ leaves $\overline{\Sigma}$ and returns to Γ_- , if it ever returns, at

a time greater than T. We may also assume that the time each solution spends in $\bar{\Sigma}$ is bounded by some $\bar{t} > 0$.

Now, let g be a C^1 local ϵ-perturbation of f at x_0 such that $\operatorname{supp}(f - g) \subset \Sigma$, and consider the differential equation (1.3). D_0 will be an absorbing set for (1.3) if Σ is sufficiently small since f and g agree on $D_0 \setminus \Sigma$. If the trajectory of a solution $y(t, y_0)$ of (1.3) never intersects Σ after a certain time, then it coincides with the trajectory of a solution to (1.1) for all sufficiently large time. Thus, for such a solution, it follows from (2.7) that

$$\frac{1}{t}\int_0^t \mu\big(B_g(y(s, y_0))\big)\, ds \le -\frac{\delta}{4}$$

where B_g is defined in (1.2) with f being replaced by g.

Suppose the trajectory of $y(t, y_0)$ intersects Σ infinitely often. We may assume that $y_0 \in \Gamma_+$. Let $t_0 = 0$ and

$$T < s_1 < t_1 < s_2 < t_2 < \cdots < s_n < t_n < \cdots$$

be such a sequence that

(i) s_i and t_i are the time $y(t, y_0)$ intersects Γ_- and Γ_+, respectively, each time it returns to Σ,

(ii) $y(t, y_0) \in \Sigma \quad s_i \le t \le t_i \quad$ for each $i \ge 1$,

(iii) $y(t, y_0) \notin \Sigma \quad t_i < t < s_{i+1} \quad$ for each $i \ge 0$.

Then we have

(iv) $t_i - s_i \le \bar{t} \quad$ for each $i \ge 1$,

(v) $s_{i+1} - t_i > T \quad$ for each $i \ge 0$.

(vi) $y(t, y_0)$ coincides with the solution $x(t, y_i)$ of (1.1) for $t_i < t < s_{i+1}$, where $y_i = y(t_i, y_0)$ for each $i \ge 0$. (See Figure 5.2.2)

Since g is a C^1 ϵ-perturbation of f, we may choose ϵ sufficiently small so that

$$|\mu(B_f(x)) - \mu(B_g(y))| < \frac{\delta}{4\bar{t}}$$

for all x, y in Σ.

Therefore, for each $i \geq 0$,

$$\int_{t_i}^{t_{i+1}} \mu\big(B_g(y(t,y_0))\big)dt = \int_{t_i}^{t_{i+1}} \mu\big(B_f(x(t,y_i))\big)dt +$$

$$\int_{t_i}^{t_{i+1}} \big[\mu\big(B_f(x(t,y_i))\big) - \mu\big(B_g(y(t,y_0))\big)\big]\, dt$$

$$\leq -\frac{\delta}{2}\big(t_{i+1}-t_i\big) + \big(t_{i+1}-s_{i+1}\big)\frac{\delta}{4\bar{t}}$$

$$\leq -\frac{\delta}{2}\big(t_{i+1}-t_i\big) + \frac{\delta}{4} \leq -\frac{\delta}{4}\big(t_{i+1}-t_i\big) \tag{2.11}$$

Thus for all sufficiently large t, assume that $t_n < t \leq t_{n+1}$ for some n, we have

$$\frac{1}{t}\int_0^t \mu\big(B_g(y(t,y_0))\big) = \frac{1}{t}\int_0^{t_n} \mu(B_g) + \frac{1}{t}\int_{t_n}^t \mu(B_g)$$

$$= \frac{1}{t}\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} \mu(B_g) + \frac{1}{t}\int_{t_n}^t \mu(B_g)$$

$$\leq -\frac{\delta}{4}\frac{1}{t}\sum_{i=1}^{n-1}(t_{i+1}-t_i) + \frac{1}{t}\int_{t_n}^t \mu(B_g)$$

$$\leq -\frac{\delta}{4}\frac{t_n}{t} + \frac{1}{t}\int_{t_n}^t \mu(B_g)$$

If $t - t_n > T$, then $\frac{1}{t}\int_{t_n}^t \mu(B_g) < -\frac{\delta}{4}\frac{t-t_n}{t}$ for a similar reason used in deriving (2.11). Therefore, in this case

$$\frac{1}{t}\int_0^t \mu(B_g) < -\frac{\delta}{2}.$$

If $t - t_n \leq T$, then $\frac{t-t_n}{t} \leq \frac{T}{t}$ and thus $\frac{t_n}{t} \geq 1 - \frac{T}{t} > \frac{1}{2}$ when t is sufficiently large. Hence in this case,

$$\frac{1}{t}\int_0^t \mu(B_g) < -\frac{\delta}{4}\frac{t_n}{t} + \frac{t-t_n}{t}\max_{x\in\overline{D}_0}\mu(B_g(x)) < -\frac{\delta}{16}.$$

Therefore, when t is sufficiently large,

$$\frac{1}{t}\int_0^t \mu\big(B_g(y(s,y_0))\big)\, ds < -\frac{\delta}{16} \qquad \text{for all } y_0 \in D_0.$$

This leads to $q(g, D_0) < 0$. Thus the lemma is proved. □

Proposition 5.2.9. *If* $q(f, D_0) < 0$. *Then the dimension of the stable manifold of any equilibrium is at least* $(n-1)$. *If an equilibrium is not isolated, then its stable manifold has dimension* $(n-1)$ *and it has a centre manifold of dimension* 1 *which contains all nearby equilibria.*

Proof. Suppose x_1 is an equilibrium. Then $x(t, x_1) = x_1$ for all $t > 0$. Thus $q(f, D_0) < 0$ implies

$$\mu\left(A\,\frac{\partial f^{[2]}}{\partial x}\,A^{-1}\right) < 0$$

at x_1, for $f(x_1) = 0$ implies $A_{f(x_1)}(x_1) = 0$. The rest of the proof is essentially the same as that of Proposition 5.2.1. □

Now we can show that the conclusions of Theorem 5.2.2 remain true if the condition (2.2) is replaced by the weaker condition $q(f, D_0) < 0$.

Theorem 5.2.10. *Suppose* D *is simply connected and* D_0 *is an absorbing set for* (1.1). *If* $q(f, D_0) < 0$, *then the following hold*

(1) *Every nonwandering point of* D *is an equilibrium;*

(2) *Every nonempty alpha or omega limit set in* D *is a single equilibrium;*

(3) *Any equilibrium in* D *is the alpha limit set of at most two distinct nonequilibrium trajectories.*

Proof. Since D is simply connected, Theorem 4.1.7 of Chapter IV implies that $q(f, D_0) < 0$ is a Bendixson Criterion. Therefore $\tau(x_0) = +\infty$ for every nonwandering point x_0 which is not an equilibrium for (1.1), by Lemma 5.2.7. It then follows from Lemma 5.2.8 that $q(f, D_0) < 0$ is robust under C^1 local perturbations of f at every nonequilibrium nonwandering point for (1.1). Thus (1) follows from Theorem 5.1.2. Proofs for (2) and (3) are exactly the same as those of (d) and (e) of Theorem 5.2.2. □

Now, using the same proof as that of Theorem 5.2.6, we can show that this weaker condition $q(f, D_0) < 0$ also has the same restrictions on the dimension of compact invariant sets as those implied under (2.2).

Theorem 5.2.11. *Assume that D is simply connected and D_0 is a bounded absorbing set for (1.1). Suppose $K \subset D$ is a compact set which is invariant with respect to (1.1). If $q(f, D_0) < 0$, then its Hausdorff dimension $\dim_H K \leq 1$. In particular, if K is also connected, then $\dim_H K = 0$ or 1 depending on whether K contains one point or more than one point.*

Remark. As we mentioned earlier in Chapter IV, when $A = I$ and $|\cdot|$ is the euclidean norm on $\mathbf{R}^N$, $q(f, D_0)$ is related to a quantity q_2 considered by R. Temam [17], with the difference being that Temam defines q_2 on a compact invariant set (e.g. the global attractor), while we define $q(f, D_0)$ on a bounded absorbing set which usually contains a neighbourhood of the global attractor. Temam proved in [17] that if $q_2 < 0$ on the global attractor A, which always exists when there is a bounded adsorbing set, then $\dim_H A \leq 2$. Here under a slightly stronger condition $q(f, D_0) < 0$, we are able to show $\dim_H A \leq 1$.

More generally, we can consider a real-valued function $(x, y) \mapsto V(x, y)$ defined for $(x, y) \in D \times \mathbf{R}^N$, $N = \binom{n}{2}$. We assume that V is locally Lipschitz continuous in its domain and

$$\lim_{h \to 0+} \frac{1}{h} [V(x + ha, y + hb) - V(x, y)]$$

exists for all $(x, y) \in D \times \mathbf{R}^N$ and all $(a, b) \in \mathbf{R}^n \times \mathbf{R}^N$. For each $(x, y) \in D \times \mathbf{R}^N$, we define $\dot{V}(x, y)$ by

$$\dot{V}(x, y) = \lim_{h \to 0+} \frac{1}{h} \left[V\Big(x + hf(x), y + h \frac{\partial f^{[2]}}{\partial x}(x) y\Big) - V(x, y) \right]. \tag{2.12}$$

Then we have seen in Theorem 3.1.3 of Chapter III that

$$\dot{V}(x, y) = \frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial y}^{*} \frac{\partial f^{[2]}}{\partial x}(x)\, y \tag{2.13}$$

almost everywhere.

Suppose D is simply connected. We have established in Theorem 4.1.3 of Chapter IV that the following condition gives rise to a Bendixson Criterion: there exist constants $a, b > 0$ such that

$$V(x,y) \geq a\,|y|, \quad \text{and} \quad \overset{\bullet}{V}(x,y) \leq -b\,|y| \tag{2.14}$$

for all $(x,y) \in D \times \mathbf{R}^N$, $N = \binom{n}{2}$. Condition (2.14) is clearly robust under C^1 local perturbations of f at all points of D. Therefore our wandering point theorem (Theorem 5.1.2) implies that every nonwandering point in D is an equilibrium. In fact, as the following result shows, all conclusions we draw from (2.2) or $q(f, D_0) < 0$ remain true under the condition (2.14).

Theorem 5.2.12. *Assume that* D *is simply connected and that* (1.1) *is dissipative. Then the conclusions in Proposition* 5.2.1, *Theorem* 5.2.2 *and Theorem* 5.2.6 *remain valid under* (2.14).

Proof. The assertions in Proposition 5.2.1 on the dimension of the stable manifold and existence of centre manifold at a equilibrium x_0 is proved by showing that the real parts of all eigenvalues of $\frac{\partial f}{\partial x}^{[2]}(x_0)$ are negative. This follows from Theorem 3.3.2 in Chapter III that (2.14) implies the second compound equation of the linear variational system of (1.1), with respect to any solution, is asymptotically stable. Conclusions in Theorem 5.2.2 and Theorem 5.2.6 can be proved under (2.14) using the original proofs without change. $\square$

§5.3. Global Stability

Throughout this section, we assume that $\bar{x}$ is the only equilibrium of (1.1) in D. Recall that an equilibrium x_0 is said to be *stable* if, for each $\epsilon > 0$, there exists $\delta > 0$ such that $|x - x_0| < \delta$ implies $|\phi_t(x) - x_0| < \epsilon$, where ϕ_t is the flow generated by (1.1); x_0 is said to *attract* a subset B of D if $\mathrm{d}(\phi_t(B), x_0) \to 0$

as $t \to \infty$; x_0 is said to be *asymptotically stable* if it is stable and attracts every point in a neighbourhood; x_0 is *uniformly asymptotically stable* if it is stable and attracts a neighbourhood. It has been established in Chapter I that, for a stable equilibrium x_0, it attracts every point in a neighbourhood if and only if it attracts a neighbourhood, namely attracts all the points in the neighbourhood uniformly. It then follows that asymptotic stability is equivalent to uniform asymptotic stability. We want to point out this equivalence of asymptotic stability and uniformly asymptotic stability in general is only true for dynamical systems in $\mathbf{R}^n$. As a matter of fact, examples exist (see [7]) to show that this equivalence does not hold for some dynamical systems in infinite dimensional spaces.

We would also like to remark that x_0 attraction of every point in a neighbourhood does not necessarily imply that x_0 is stable. This can be demonstrated by an equilibrium with a homoclinic orbit (Figure 5.3.1). Concrete examples to show this can be found in [7] and [9].

The *basin of attraction* of x_0 is the union of all the points which are attracted by x_0 . If x_0 is asymptotically stable, then it follows from the continuous dependence on initial values that its basin of attraction is an open subset of D . If x_0 is a stable equilibrium and its basin of attraction is D, then we say that x_0 is *globally asymptotically stable*. Sometimes this is also called *asymptotically stable in the large.*

The problem of global stability is very interesting and very important in the analysis of nonlinear systems of differential equations which model some natural phenomena. In Chapter VII, we will see how this problem arises in a model from Epidemiology. It is well known that this problem is also very difficult to solve. The only approach in full generality which is widely used thus far is construction of Lyapunov functions (or functionals). The following theorem is a typical result of this type, a proof of which can be found in [6] or [7].

Theorem 5.3.1. *We assume that* (1.1) *is dissipative and* $\bar{x}$ *is the only equilibrium in* D. *Suppose there exists a* C^1 *function* $V(x)$ *which satisfies the following:*

(a) $V(x) > 0$ *for all* $x \in D$, *and* $= 0$ *only when* $x = \bar{x}$,

(b) $\overset{\bullet}{V}(x) = \frac{\partial V}{\partial x}^* f(x) \leq 0$ *for all* $x \in D$ *and* $= 0$ *only when* $x = \bar{x}$.

Then $\bar{x}$ *is globally asymptotically stable.*

Remarks.

(i). The smoothness of V can be weakened so that V is locally Lipschitz and

$$\lim_{n \to 0+} \frac{1}{h} \left[V(x + hf(x)) - V(x) \right] \tag{3.1}$$

exists for all $x \in D$.

(ii). If conditions (a) and (b) only hold in a neighbourhood of $\bar{x}$, then $\bar{x}$ is locally asymptotically stable.

Theorem 5.3.1 has many variations, but most of them require that V be positive definite in some sense as in the condition (a) of this theorem. We will see in the following that this positive definiteness assumption can be dropped. A More general result is formulated in §5.4 in the discussion of LaSalle's Invariance Principle.

Proposition 5.3.2. *Suppose there exists a* C^1 *function* $V(x)$ *such that*

$$\overset{\bullet}{V}(x) =: \frac{\partial V}{\partial x}^* f(x) < 0 \qquad \text{whenever } f(x) \neq 0. \tag{3.2}$$

Then every nonwandering point is an equilibrium.

Proof. Since (3.2) implies that every periodic orbit has to stay in the set where $\overset{\bullet}{V}(x) = 0$, (4.2) is actually a Bendixson's Criterion, which is also robust under local C^0 perturbations of f at every nonequilibrium point of D. Hence the proposition follows from Theorem 5.1.2. □

Theorem 5.3.3. *Suppose that* $D_1 \subset D$ *is a positively invariant subset and* $\bar{x}$ *is the only equilibrium in* D_1. *If* (3.2) *holds in* D_1 , *then* $\bar{x}$ *is globally asymptotically stable in* D_1 .

Proof. The positive invariance of D_1 and Proposition 5.3.2 imply that $\omega(x_0) = \bar{x}$ for all $x_0 \in D_1$. Therefore $\bar{x}$ attracts every point of D_1. It remains to show that $\bar{x}$ is locally stable. Since V strictly decreases along any nonequilibrium trajectory, we know $V(\bar{x}) \leq V(x)$ for all $x \in D_1$ and the equality holds only when $x = \bar{x}$. Now the new function $V_1 = V - V(\bar{x})$ satisfies conditions (a) and (b) of Theorem 5.3.1. Therefore $\bar{x}$ is globally asymptotically stable in D_1. □

Now suppose D_0 is an absorbing set and (3.2) holds on D_0. Since every trajectory in D eventually enters and remains in D_0, we can conclude from Proposition 5.3.2 that $\bar{x}$ attracts every point in D. A similar argument to that in the proof of Theorem 5.3.3 shows that conditions (a) and (b) hold on D_0 which implies that $\bar{x}$ is locally stable from the remark (ii) following Theorem 5.3.1. Therefore $\bar{x}$ is globally asymptotically stable in D. We thus have the following global version of the above theorem.

Theorem 5.3.4. *Assume that* (1.1) *is dissipative. Suppose* (3.2) *holds in* D *and* $\bar{x}$ *is the only equilibrium. Then* $\bar{x}$ *is globally asymptotically stable in* D.

Remarks.

(i). The conclusion in the above theorem remains true if (3.2) is only assumed to hold on a bounded absorbing set $D_0 \subset D$.

(ii). If $D = \mathbf{R}^n$, instead of assuming that (1.1) is dissipative, we can impose the following condition

$$V(x) \to \infty \qquad \text{as} \quad |x| \to \infty.$$

and the theorem still holds.

(iii). Proposition 5.3.3 and Theorem 5.3.4 may also be proved using the 'Invariance Principle' of LaSalle (see[11]). We will discuss this in more detail later in the §5.4 of this chapter.

Corollary 5.3.5. *Assume that* D *is simply connected and that* $D_0 \subset D$ *is a bounded absorbing set and contains a unique equilibrium* $\bar{x}$. *Suppose there exists a* C^1 *function* $x \to a(x) \in \mathbf{R}^n$ *such that*

(1) $a^* f(x) < 0$, *if* $f(x) \neq 0$, $x \in D_0$,

(2) $\frac{\partial a_i}{\partial x_j} - \frac{\partial a_j}{\partial x_i} = 0$, *in* D_0 *for all* $i, j = 1, \cdots, n$.

Then $\bar{x}$ *is globally asymptotically stable.*

Proof. This follows the fact that (2) implies $\frac{\partial V}{\partial x} = a$ for some scalar function V and then (1) shows that (3.2) is satisfied in D_0. □

Remark. The conditions of Corollary 5.3.5 are satisfied if f satisfies $\frac{\partial f_i}{\partial x_j} - \frac{\partial f_j}{\partial x_i} = 0$, $i, j = 1, \cdots, n$, since we may then choose $\alpha = -f$. Then, for example $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$ is a 'Bendixson Criterion' for the 2-dimensional system $x' = P(x, y)$, $y' = Q(x, y)$. The result may be considered as saying that vector fields near one which is irrotational have no periodic orbits.

One drawback of results like Theorem 5.3.4 is that the function V (often called Lyapunov function after the Russian mathematician who invented the idea) is hard to construct and that different forms of f need different treatments. In the rest of the section, we present some concrete criteria for global stability which can be easily computed from the equation (1.1). These criteria are based on the autonomous convergence theorems discussed in §5.2.

Let $(x, y) \mapsto V(x, y)$ be a locally Lipschitz continuous real-valued function defined for $(x, y) \in \mathbf{R}^n \times \mathbf{R}^N$, $N = \binom{n}{2}$ considered in §5.2, and $\dot{V}(x, y)$ be as defined in (2.12). Then

$$\dot{V}(x, y) = \frac{\partial V^*}{\partial x} f(x) + \frac{\partial V^*}{\partial y} \frac{\partial f}{\partial x}(x) y$$

almost everywhere. Using Theorem 5.2.12, we can prove the following result on global stability.

Theorem 5.3.6. *Assume that D is simply connected and that (1.1) is dissipative and has a unique equilibrium $\bar{x}$ in D. Suppose there exist constants $a, b > 0$ such that*

(1) $V(x,y) \geq a|y|$,

(2) $\overset{\bullet}{V}(x,y) \leq -b|y|$

for almost all $(x,y) \in D \times \mathbf{R}^N$, $N = \binom{n}{2}$. Then $\bar{x}$ is globally asymptotically stable in D.

Proof. Clearly $\{x_0\}$ is globally attracting since Theorem 5.2.12 implies it is the omega limit set of every trajectory. Moreover it is stable since otherwise it would be both the alpha limit set and the omega limit set of some homoclinic trajectory $\gamma = \{x(t) : t \in (-\infty, \infty)\}$. In this circumstance we assert that $\mathcal{C} = \gamma \cup \{x_0\}$ is the trace of a rectifiable simple closed curve. This curve is invariant with respect to (1.1), $x(t, \mathcal{C}) = \mathcal{C}$, and existence of such an invariant curve is not possible under the assumptions of the theorem by Theorem 4.1.3 of Chapter IV. It remains to prove that $\mathcal{C}$ is rectifiable. Since γ is in the C^1 centre manifold or unstable manifold of x_0, and this is 1-dimensional, it is only necessary to show that $\gamma_+ = \{x(t) : t \in [1, \infty)\}$ is the trace of a rectifiable curve. This is rectifiable if it also approaches its omega limit through a 1-dimensional centre manifold. If it does not approach x_0 through a centre manifold then, by the Centre Manifold Theorem, it approaches x_0 exponentially in time. Thus $|f(x(t))| \leq Ce^{-\lambda t}$ for some constants C, $\lambda > 0$, since $f(x_0) = 0$. Considering $\tau(s) = (1-s)^{-1}$, $y(s) = x(\tau(s))$, $s \in [0,1)$, we find $y'(s) = f(x(\tau(s)))\tau'(s)$ so that $|y'(s)| \leq Ce^{-\lambda\tau(s)}\tau'(s) = Ce^{-\lambda(1-s)^{-1}}(1-s)^{-2}$ and y' is bounded with $y[0,1) = \gamma_+$. □

Remark. From the above proof we can also draw an important conclusion: under the conditions (1) and (2) in Theorem 5.3.6, the simple closed invariant curves arising from the following types of orbits of (1.1) are all rectifiable,

(i) homoclinic orbits;

(ii) a pair of heteroclinic orbits connecting the same two equilibria;

(iii) heteroclinic cycles.

(see Figure 4.1.1).

Therefore, as we have seen in Theorem 4.1.3 of Chapter IV, when D is simply connected and (1.1) is dissipative, existence of periodic orbits as well as orbits of types (i) (ii) (iii) is precluded by the conditions (1) and (2) in Theorem 5.3.6. As we will see in the following discussion, these orbits can not exist under the assumptions of Theorems 5.3.7, 5.3.9 and Corollary 5.3.8.

As is discussed in previous chapters, a general form of the function V which is interesting to us is given by

$$V(x,y) = |A(x)\,y|$$

where $x \to A(x)$ is a $N \times N$ matrix-valued function nonsingular and C^1 in D, and $|\cdot|$ is a vector norm in $\mathbf{R}^N$, $N = \binom{n}{2}$. Let μ be the corresponding Lozinskiĭ measure. Then the assumptions on A and the following general condition of Dulac type

$$\mu\big(A_f A^{-1} + A\frac{\partial f}{\partial x}^{[2]} A\big) < 0 \qquad \text{in} \quad D \tag{3.4}$$

implies V satisfies the conditions (1) and (2) in Theorem 5.3.6. Therefore we have the following result.

Theorem 5.3.7. *Assume that D simply connected and (1.1) is dissipative and has a unique equilibrium $\bar{x}$. Suppose (3.4) is satisfied. Then $\bar{x}$ is globally asymptotically stable in D.*

Remark. Theorem 5.3.7 remains valid if (3.4) only holds on a bounded absorbing set $D_0 \subset D$.

With $A = I$, Theorem 5.3.7 gives rise to the following corollary.

Corollary 5.3.8. *Under the assumptions of Theorem 5.3.7, $\bar{x}$ is globally asymptotically stable if one of*

$$\mu\Big(\frac{\partial f}{\partial x}^{[2]}\Big) < 0, \qquad \mu\Big(\frac{\partial f}{\partial x}^{[2]}\Big) < 0 \tag{3.5}$$

holds in D.

Remark. When the Lozinskiĭ measure μ is calculated corresponding to the l_1, l_∞ and l_2 norms on $\mathbf{R}^N$, (3.5) gives rise to the following concrete conditions each imply the global asymptotic stability of $\bar{x}$ under the assumptions of Theorem 5.3.7

(i) $\sup\left\{ \frac{\partial f_r}{\partial x_r} + \frac{\partial f_s}{\partial x_s} + \sum_{q\neq r,s} \left(\left|\frac{\partial f_q}{\partial x_r}\right| + \left|\frac{\partial f_q}{\partial x_s}\right| \right) \ : 1 \leq r < s \leq n \right\} < 0,$

(ii) $\sup\left\{ \frac{\partial f_r}{\partial x_r} + \frac{\partial f_s}{\partial x_s} + \sum_{q\neq r,s} \left(\left|\frac{\partial f_r}{\partial x_q}\right| + \left|\frac{\partial f_s}{\partial x_q}\right| \right) \ : 1 \leq r < s \leq n \right\} < 0,$

(iii) $\lambda_1 + \lambda_2 < 0,$

(iv) $\inf\left\{ \frac{\partial f_r}{\partial x_r} + \frac{\partial f_s}{\partial x_s} - \sum_{q\neq r,s} \left(\left|\frac{\partial f_q}{\partial x_r}\right| + \left|\frac{\partial f_q}{\partial x_s}\right| \right) \ : 1 \leq r < s \leq n \right\} > 0,$

(v) $\inf\left\{ \frac{\partial f_r}{\partial x_r} + \frac{\partial f_s}{\partial x_s} - \sum_{q\neq r,s} \left(\left|\frac{\partial f_r}{\partial x_q}\right| + \left|\frac{\partial f_s}{\partial x_q}\right| \right) \ : 1 \leq r < s \leq n \right\} > 0,$

(vi) $\lambda_{n-1} + \lambda_n > 0.$

where $\lambda_1 \geq \cdots \geq \lambda_n$ are eigenvalues of the symmetric matrix $\frac{1}{2}(\frac{\partial f}{\partial x}^* + \frac{\partial f}{\partial x})$.

The following quantities are defined in (2.6) and (2.7) of §5.2,

$$q(f, D_0, t) = \sup_{x_0 \in \bar{D}_0} \frac{1}{t} \int_0^t \mu\big(B(x(\tau, x_0))\big)\, d\tau$$

$$q(f, D_0) = \limsup_{t\to\infty} q(D_0, t)$$

where

$$B = A_f A^{-1} + A \frac{\partial f}{\partial x}^{[2]} A^{-1}$$

as is given in (2.5). In the following, we will omit f from these notations, since there will be no ambiguity.

An autonomous convergence theorem (Theorem 5.2.10) is proved under the condition $q(D_0) < 0$. Moreover, $q(D_0) < 0$ also implies that the unique equilibrium $\bar{x}$ is locally stable since otherwise $\bar{x}$ will be both the alpha and omega limit set of a homoclinic orbit Γ. Now the simple closed invariant curve $\Gamma \cup \{\bar{x}\}$ is rectifiable under the condition $q(D_0) < 0$ for the same reason as given in the proof of Theorem 5.3.6, and thus is precluded by $q(D_0) < 0$ if D is simply connected. Therefore, $\bar{x}$ is globally asymptotically stable under this condition. This yields the following result on global stability under a weaker condition than (3.3).

Theorem 5.3.9. *Assume that D is simply connected and that (1.1) is dissipative with a bounded absorbing set D_0. Suppose $\bar{x}$ is the unique equilibrium of (1.1). Then $\bar{x}$ is globally asymptotically stable provided $q(D_0) < 0$.*

Applications of these global stability results will be seen in Chapter VII.

§5.4 Weak Dulac Conditions and LaSalle's Invariance Principle

Let $x \to V(x) \in R$ be a real-valued C^1 function defined for $x \in D$. Define, for each $x \in D$,

$$\overset{\bullet}{V}(x) = \frac{\partial V}{\partial x} f(x). \tag{4.1}$$

Suppose G is a subset of D. A C^1 function V is said to be a *Lyapunov function* on G if $\overset{\bullet}{V}(x) \leq 0$ for all $x \in G$. Let

$$E = \{x \in \bar{G} \cap D \; : \; \overset{\bullet}{V}(x) = 0\} \tag{4.2}$$

and M be the largest invariant set in E. The following result is a version of LaSalle's Invariance Principle [11].

Theorem 5.4.1. *Suppose V is a Lyapunov function on G and $\gamma^+(x_0)$ is a bounded semi-orbit of (1.1) which lies in G. Then the ω-limit set of $\gamma^+(x_0)$ belongs to M; that is $x(t, x_0) \to M$ as $t \to \infty$.*

We have mentioned earlier that Theorem 5.3.3 and Theorem 5.3.4 can be derived from this Invariance Principle. In fact condition (3.2) implies V is a Lyapunov function on D_1. Since $\bar{x}$ is the only equilibrium, $M = \{\bar{x}\}$. Therefore $\bar{x}$ attracts every point. The local stability of $\bar{x}_0$ can be proved as before.

Compared with Theorem 5.4.1 or results of similar type, the Invariance Principle has the advantage of providing information on the limit sets with less restrictive conditions. It has proved to be very useful in the analysis of global stability. LaSalle's Invariance Principle is usually stated for functions V which are only Lipschitz continuous. However, as the following result shows, stronger conclusions can be drawn when V is a smooth function.

Theorem 5.4.2. *Suppose $V \in C^1(D \to \mathbf{R})$ is a Lyapunov function on G. Then all nonwandering points of* (1.1) *in G are contained in M .*

Proof. Observe that, if x is not an equilibrium and $\dot{V}(x) < 0$, all trajectories depart from a small neighbourhood of x and, since V strictly decreases along trajectories in a neighbourhood of x, no trajectory returns. Therefore all nonwandering points in G are contained in E. Moreover, since the set of all nonwandering points is closed and invariant, the theorem is proved. □

Next we will examine what can be concluded under the following weak Dulac condition.

$$\mu\left(A_f A^{-1} + A\frac{\partial f^{[2]}}{\partial x} A^{-1}\right) \leq 0 \qquad \text{in} \quad D. \tag{4.4}$$

Let D_1 be the subset of D where (4.4) is strict and $D_2 = D \setminus D_1$.

Theorem 5.4.3. *Suppose D is simply connected, and there is a matrix A of class C^1 such that either (a) or (b) holds:*

(a) *D has the minimum property with respect to S and (5.4) is satisfied on D.*

(b) *D_0 is an absorbing compact subset of D and* (5.4) *is satisfied on D_0.*

Then the conclusions (*c*), (*d*), (*e*) *of Theorem* 5.2.2 *hold if in those statements* D *is replaced by* D_1.

This may be seen by again supposing that $x_0 \in D_1$ is nonwandering and $f(x_0) \neq 0$. Then, choosing the neighbourhood U of x_0 in Lemma 5.2.1 a sufficiently small subset of the region where (4.4) is strict, we find a system C^1-close to (1.1) which also satisfies (4.4), strictly near x_0 and has a nontrivial periodic trajectory intersecting U. Then, using Criteria I, II of Chapter IV in cases (a), (b) respectively, we find a contradiction as before. Thus we must have $f(x_0) = 0$ if $x_0 \in D_1$ is nonwandering.

The Corollaries 5.2.3, 5.2.4, 5.2.5 may also be modified in this way. Similarly the considerations leading to Theorem 5.2.6, suitably altered, lead to a modification of that result.

Theorem 5.4.4. *Suppose* D *is simply connected,*

(*a*) f *satisfies condition* (*b*) *of Theorem* 5.4.3,

(*b*) D_1 *is invariant and*

(*c*) *at most a finite number of trajectories in* $K \cap D_1$ *have a limit point in* D_2.

Then $\dim_H(K \cap D_1) \leq 1$.

This result is established by applying the proof of Theorem 5.2.6 to the complement in $K \cap D_1$ of the at most 1-dimensional set in $K \cap D_1$ consisting of those trajectories with limit points in D_2.

The condition (c) may of course be difficult to establish in general and may require the use of more than one functional S or function V or extensions of the ideas considered here.

Remark. Theorems 5.2.6, 5.3.7, 5.4.4 are somewhat surprising in view of estimates on Hausdorff dimensions of attractors due to Smith [16], Temam [17], Boichenko and Leonov [1], Eden, Foias and Temam [4]. These results, which seem to give good estimates on attractors whose dimension is greater than 2 even in delicate cases such

as the Lorenz system, would only give 2 as an upper bound in some cases where Theorems 5.2.6, 5.4.4 would give a bound 1 and Theorem 5.3.7 would give a value 0. For example, in the terminology of [4], conditions such as (2.2) would imply that $\mu_1(x) + \mu_2(x) < 0$, where $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ are the local Lyapunov exponents which in turn implies that any invariant compact set has Hausdorff dimension less than 2.

Example. Consider the dissipative system

$$\frac{dx_1}{dt} = x_1 - x_1^3, \quad \frac{dx_2}{dt} = -x_2 \tag{4.5}$$

of Eckmann and Ruelle cited by Eden [3]. Here the global attractor is $K = [-1,1] \times \{0\}$ so that $\dim_H K = 1$.

If $V(x) = x_2^2$, we find $\frac{\partial V}{\partial x}^* f(x) = -2x_2^2$ so we could infer independently from this that K has dimension at most 1 since Theorem 5.4.2 implies it is located on the x_1-axis. Since there is more than one equilibrium, we conclude $\dim_H K = 1$. Alternatively consider $A(x) = x_2^2 + 1$ so that

$$A_f A^{-1} + A \frac{\partial f}{\partial x}^{[2]} A^{-1} = -2x_2^2(x_2^2+1)^{-1} - 3x_1^2$$

and the weak Dulac condition (4.4) is satisfied with $D_2 = \{(0,0)\}$. At most a finite number of trajectories can have limit points in D_2 since otherwise we could find an invariant rectifiable simple closed curve which cannot exist by Criterion II of Chapter IV. Theorem 5.4.4 therefore implies $\dim_H K \leq 1$.

§5.5. Bibliography for Chapter V

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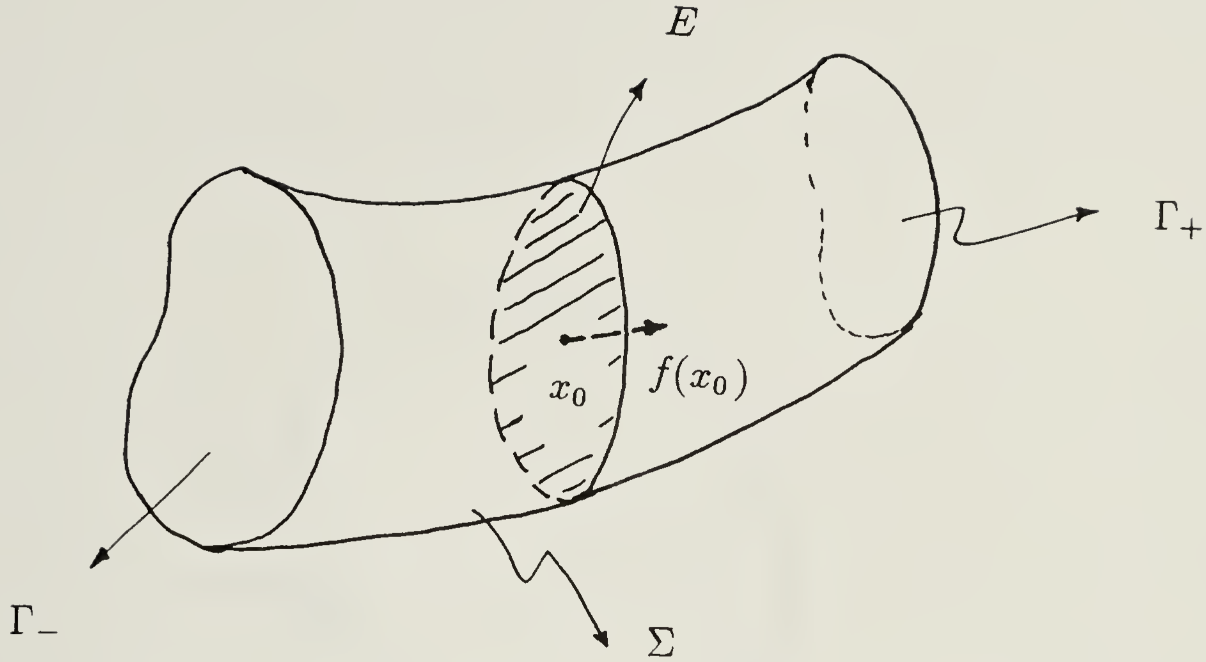

Figure 5.2.1. A flow box Σ.

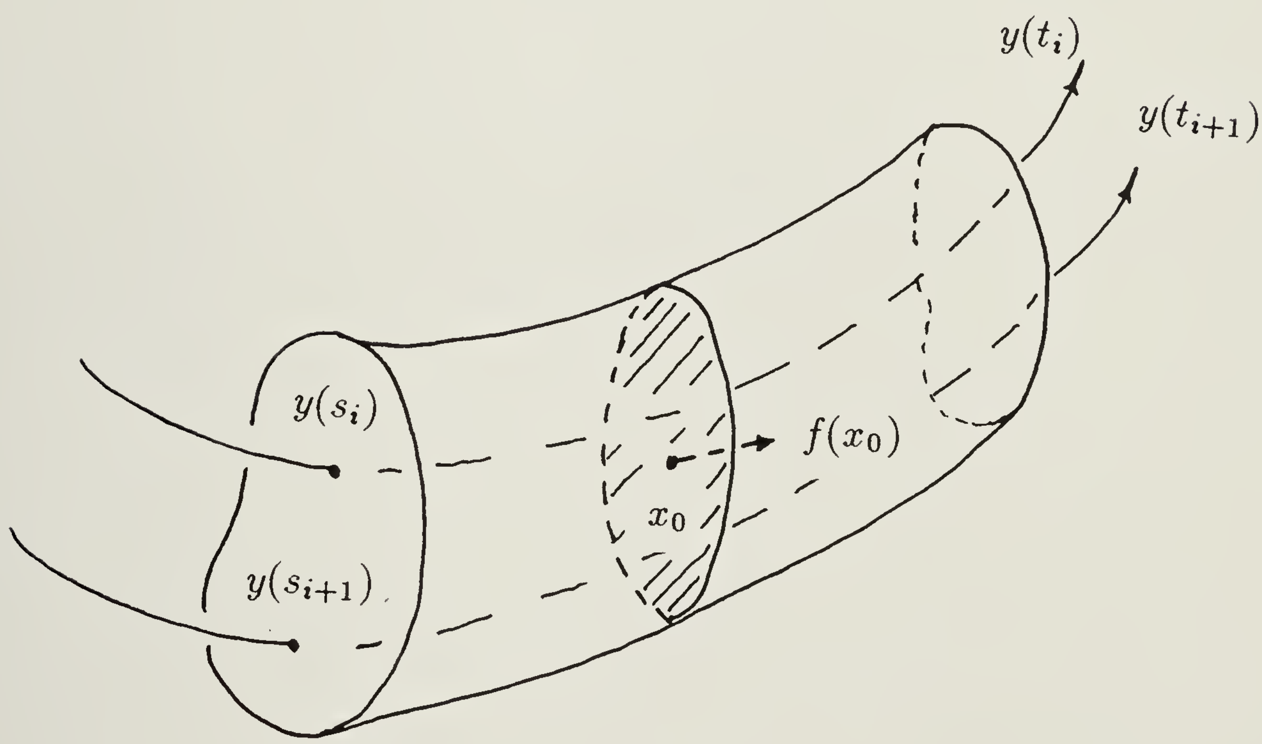

Figure 5.2.2. A trajectory intersecting Σ many times.

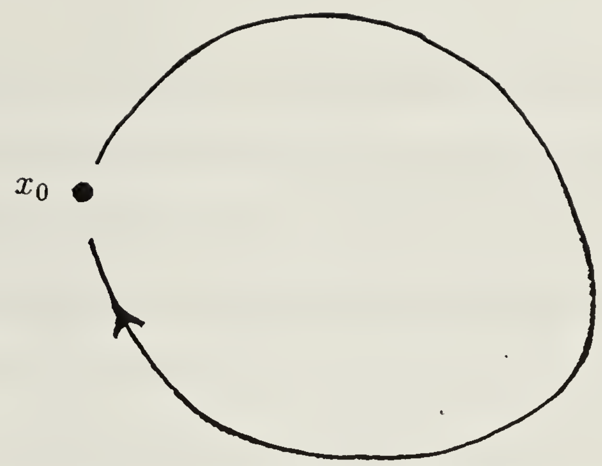

Figure 5.3.1. A fixed point with a homoclinic orbit.

CHAPTER VI

AUTONOMOUS SYSTEMS WITH FIRST INTEGRALS

In this chapter, we consider the nonlinear problems discussed in earlier chapters in the context of autonomous systems possessing first integrals. Existence of first integrals often indicates the presence of certain physical conservation laws manifested in the system. Mathematically speaking, a first integral is a nonconstant real-valued function $H(x)$ which is constant along each solution. Thus every solution stays on a lower dimensional invariant manifold defined by an equation $H(x) = C$ with C being determined by its initial value. This means that systems in $\mathbf{R}^n$ having first integrals are only capable of displaying behaviour which is typical of systems in lower dimensions. We prove that, for such systems, the results obtained in earlier chapters can now be proved under considerably less restrictive conditions. For example, the Bendixson Criterion $\mu(\frac{\partial f}{\partial x}^{[2]})$ for general systems in Chapter IV can be replaced by $\mu(\frac{\partial f}{\partial x}^{[r+2]})$ if the system has r independent first integrals. This result will play an important role in Chapter VII where we resolve some hitherto unsolved problems in Mathematical Biology.

A traditional approach to systems with first integrals is to employ the first integrals to reduce the number of variables, and thus reduce the dimension of the problem. This often relies critically on the choice of coordinates in the invariant manifold. In our study, the focus is on the implications to the linear variational equations. This leads to the discovery of a nice geometric characterization (Theorem 6.4.2) which is crucial to the development. Then techniques from exterior algebra are used to establish the main result for linear theory (Theorem 6.1.6). This enables us to develop nonlinear results by the methods discussed in earlier chapters.

After the linear theory is established in §6.1, we deal with the case when the first integrals are affine in sections §6.2 and §6.3. The general case is treated in the

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final section §6.4.

§6.1. Linear Systems with an Invariant Subspace

In this section, we discuss a special class of linear systems of ordinary differential equations which leave a subspace of $\mathbf{R}^n$ invariant. The linear variational equations of many nonlinear autonomous systems which we will study in the following sections belong to this class. We will see that behaviour of these special linear systems determines the geometry of the corresponding nonlinear systems. Hence results in this section are essential to the study of the asymptotic behaviour of solutions to the nonlinear systems in the following sections.

Let $\langle\ ,\ \rangle$ and $\|\cdot\|$ denote the euclidean inner product and norm, respectively, and $t \mapsto A(t)$ be a $n \times n$ matrix-valued function continuous in $\mathbf{R}$. We consider the linear system of n differential equations

$$x'(t) = A(t)\,x(t) \tag{1.1}$$

subject to the condition that there exists a constant matrix B, such that

$$B\,A(t) = 0, \qquad \text{for all} \quad t \in \mathbf{R}. \tag{1.2}$$

We denote the kernel of B by V_0 and its orthogonal complement in $\mathbf{R}^n$ by $V_0^{\perp}$. Then $\mathbf{R}^n = V_0 \oplus V_0^{\perp}$ and $V_0^{\perp} \cong \operatorname{Im} B^*$ (see [13] Theorem 12.10), where the asterisk denotes the transposition. Moreover, if $\operatorname{rank} B = r$, then $\dim V_0 = n - r$, $\dim V_0^{\perp} = r$.

Let $\mathcal{X}$ be the solution space of (1.1) and $\mathcal{X}_0$ be the subspace of $\mathcal{X}$ consisting of those solutions $x = x(t)$ of (1.1) with $x(t_0) \in V_0$, for some $t_0 \in \mathbf{R}$. The subspace V_0 of $\mathbf{R}^n$ is said to be *invariant* with respect to (1.1) if $x(t) \in V_0$ for all $t \in \mathbf{R}$, when $x = x(t)$ is a solution in $\mathcal{X}_0$.

Theorem 6.1.1. *Suppose* (1.2) *is satisfied. Then* V_0 *is invariant with respect to* (1.1).

Proof. From (1.1) and (1.2) we have $(Bx(t))' = BA(t)x(t) \equiv 0$. Hence, for every solution $x = x(t)$ of (1.1), $Bx(t) = Bx(t_0)$ for all $t, t_0 \in \mathbf{R}$, which leads to the conclusion of the theorem. □

Remark. In the rest of this section, our interest will primarily be in the behaviour of solutions in $\mathcal{X}_0$, or equivalently speaking, we will study (1.1) restricted to the invariant subspace V_0. We will see later that the need for this consideration arises in the following sections.

For vectors $u_1, \ldots, u_k \in \mathbf{R}^n$, $u_1 \wedge \cdots \wedge u_k$ denotes their *exterior product*, which is a vector in $\wedge^k \mathbf{R}^n \cong \mathbf{R}^N$, $N = \binom{n}{k}$. An inner product and the corresponding norm can be defined canonically in $\wedge^k \mathbf{R}^n$ from those of $\mathbf{R}^n$ (see Appendix B). We will also denote them by $\langle\ ,\ \rangle$ and $\|\cdot\|$, respectively, for simplicity of notation. We need the following property of this canonical inner product in $\wedge^k \mathbf{R}^n$.

Lemma 6.1.2. *Suppose* $u_1, \ldots, u_k, v_1 \ldots, v_m \in \mathbf{R}^n$, *and* $\langle u_i, v_j \rangle = 0$, $1 \le i \le k$, $1 \le j \le m$. *Let* $\Delta = u_1 \wedge \cdots \wedge u_k, \Lambda = v_1 \wedge \cdots \wedge v_m$. *Then*

$$\langle \Delta \wedge y, x \wedge \Lambda \rangle = \langle \Delta, x \rangle \langle y, \Lambda \rangle \tag{1.3}$$

for all $x \in \wedge^k \mathbf{R}^n$ *and* $y \in \wedge^m \mathbf{R}^n$.

Proof. Since each element in $\wedge^k \mathbf{R}^n$ is a linear combination of terms like $e_1 \wedge \cdots \wedge e_k$, we may assume that $x = u_1' \wedge \cdots \wedge u_k'$ and $y = v_1' \wedge \cdots \wedge v_m'$; the general case can be proved using the bilinearity of $\langle\ ,\ \rangle$. By definition

$$\langle \Delta \wedge y, x \wedge \Lambda \rangle = \det \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}_{(k+m)\times(k+m)}, \tag{1.4}$$

where D_{ij} are blocks given by

$$D_{11} = (\langle u_i, u_j' \rangle)_{k\times k}, \qquad D_{12} = (\langle u_i, v_j \rangle)_{k\times m},$$

$$D_{21} = (\langle v_i', u_j' \rangle)_{m\times k}, \qquad D_{22} = (\langle v_i', v_j \rangle)_{m\times m}.$$

Observe that $\det D_{11} = \langle \Delta, x \rangle$, $\det D_{22} = \langle y, \Lambda \rangle$ by definition, and $D_{12} = 0$ from the assumptions, the lemma is proved by expanding the determinant in (1.4). □

Let $X(t)$ be the fundamental solution of (1.1) with $X(t_0) = I_{n\times n}$, for some $t_0 \in \mathbf{R}$, where $I_{n\times n}$ is the $n \times n$ identity matrix. The following result is essential to all developments in this section.

Proposition 6.1.3. *Assume that the system* (1.1) *satisfies* (1.2). *If* $u \in \mathbf{R}^n$, *and* $w \in V_0^{\perp}$. *Then*

$$\langle X(t)u, w\rangle = \langle u, w\rangle \qquad \text{for all} \quad t \in \mathbf{R}. \tag{1.5}$$

Proof. Since $V_0^{\perp} \cong \operatorname{Im} B^*$, we can assume $w = B^* v$, for some v. Now

$$\frac{d}{dt}\langle X(t)u, w\rangle = \langle A(t)X(t)u, w\rangle = \langle BA(t)X(t)u, v\rangle = 0$$

for all $t \in \mathbf{R}$, hence $\langle X(t)u, w\rangle = \langle X(t_0)u, w\rangle = \langle u, w\rangle$. □

In the following, for a $n \times n$ matrix A, and integer $1 \le k \le n$, we use $A^{[k]}$ to denote the *k-th additive compound matrix of* A. This is a $N \times N$ matrix, $N = \binom{n}{k}$. A survey on the definition and properties of additive compound matrices together with their connections to differential equations is provided in Appendix B. Here we only mention a few properties that will be used in this chapter. We refer the readers to the Appendix B for their proof.

The term *additive* comes from the property $(A + B)^{[k]} = A^{[k]} + B^{[k]}$; if $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A, then all the possible sums of form $\lambda_{i_1} + \cdots + \lambda_{i_k}$, $1 \le i_1 < \cdots < i_k \le n$, give the eigenvalues of $A^{[k]}$; in the two extreme cases when $k = 1$ and n, we have

$$A^{[1]} = A \quad \text{and} \quad A^{[n]} = \operatorname{tr}(A), \tag{1.6}$$

respectively.

The connection between additive compound matrices and linear systems of ordinary differential equations can be seen from the following proposition, a proof of which can be found in the Appendix B.

Proposition 6.1.4. *Suppose* $x_1(t), \ldots, x_k(t)$ *are solutions of* (1.1). *Then* $y(t) = x_1(t) \wedge \cdots \wedge x_k(t)$ *is a solution of the linear system*

$$y'(t) = A^{[k]}(t)\, y(t). \tag{1.7}$$

Equation (1.7) is called the *k-th compound equation* of (1.1). When $k = 1$ and n, as a result of (1.6), (1.7) becomes the original system (1.1) and the well-known Liouville equation (see [5], Chapter V), respectively. To see the latter, recall that $u_1 \wedge \cdots \wedge u_n = \det\,(u_1, \ldots, u_n)$ for any n vectors $u_1, \ldots, u_n$ in $\mathbf{R}^n$, where $(u_1, \ldots, u_n)$ denotes the $n \times n$ matrix with the i-th column given by the coordinate vector of u_i.

Let $u_i, \cdots, u_k$ be k linearly independent vectors in $\mathbf{R}^n$. Then $X(t)u_1, \cdots, X(t)u_k$ are k linearly independent solutions to (1.1). Let $\{w_1, \cdots, w_r\}$ be an orthonormal basis of the subspace $V_0^{\perp}$ and set

$$\Omega(t) =: X(t)u_1 \wedge \cdots \wedge X(t)u_k.$$

Then $y(t) = \Omega(t)$ and $z(t) = X(t)w_1 \wedge \cdots \wedge X(t)w_r \wedge \Omega(t)$ are solutions to (1.7) and following linear system

$$z'(t) = A^{[k+r]}(t)\, z(t), \tag{1.8}$$

respectively. The following relation is important.

Proposition 6.1.5. *For any* k *elements* $u_1, \cdots, u_k$ *in* V_0,

$$\|\Omega(t)\| \leq \|X(t)w_1 \wedge \cdots \wedge X(t)w_r \wedge \Omega(t)\|, \tag{1.9}$$

for all $t \in \mathbf{R}$.

Proof. If $u_1, \ldots, u_k$ are linearly dependent, so are $X(t)u_1, \ldots, X(t)u_k$, from the uniqueness of solutions of the linear system (1.1). This leads to $\Omega(t) := X(t)u_1 \wedge \cdots \wedge X(t)u_k \equiv 0$; the proposition holds trivially. Now assume that $u_1, \ldots, u_k$ are linearly independent, so that $\Omega(t) \neq 0$ for all $t \in \mathbf{R}$. We claim the following: for all $t \in \mathbf{R}$,

(1) $\langle X(t)w_1 \wedge \cdots \wedge X(t)w_r, w_1 \wedge \cdots \wedge w_r \rangle = 1$,

(2) $\langle X(t)w_1 \wedge \cdots \wedge X(t)w_r \wedge \Omega(t), w_1 \wedge \cdots \wedge w_r \wedge \Omega(t) \rangle = \|\Omega(t)\|^2$,

(3) $\|w_1 \wedge \cdots \wedge w_r \wedge \Omega(t)\| = \|\Omega(t)\|$,

Observe that $\langle X(t)w_i, w_j \rangle = \langle w_i, w_j \rangle = \delta_{ij}$, $1 \leq i, j \leq r$, for all $t \in \mathbf{R}$, by Proposition 6.1.3. Thus $\langle X(t)w_1 \wedge \cdots \wedge X(t)w_r, w_1 \wedge \cdots \wedge w_r \rangle = \langle X(t)w_1, w_1 \rangle \cdots \langle X(t)w_r, w_r \rangle$ $= 1$, from the definition of the inner product in $\wedge^r \mathbf{R}^n$. Hence (1) follows.

To show (2), observe $\langle X(t)u_i, w_j \rangle = \langle u_i, w_j \rangle = 0$, for all $t \in \mathbf{R}, i = 1, \ldots, k$, and $j = 1, \ldots, r$. Then the identity follows from choosing $\Delta = X(t)w_1 \wedge \cdots \wedge X(t)w_r, \Lambda = y = \Omega(t), x = w_1 \wedge \cdots \wedge w_r$ in Lemma 6.1.2, and using (1). Identity (3) can be proved in the same way.

Using the Schwarz inequality in (2), we have

$$\|\Omega(t)\|^2 \leq \|X(t)w_1 \wedge \cdots \wedge X(t)w_r \wedge \Omega(t)\| \cdot \|w_1 \wedge \cdots \wedge w_r \wedge \Omega(t)\|.$$

The inequality in (1.9) now follows from (3) and the fact $\|\Omega(t)\| \neq 0$ for all $t \in \mathbf{R}$. □

Now suppose that the linear system (1.8) is asymptotically stable. Then $X(t)w_1 \wedge \cdots \wedge X(t)w_r \wedge \Omega(t) \to 0$ as $t \to \infty$, which implies, by (1.9), that $\Omega(t) \to 0$ as $t \to \infty$. We thus have the following result which is an attempt to study the stability of (1.1) when restricted to the invariant subspace V_0.

Theorem 6.1.6. *Assume that the system* (1.1) *satisfies* (1.2), *and* $\operatorname{rank} B = r$. *Then for any* $u_1, \ldots, u_k \in V_0$, $\lim_{t\to\infty} X(t)u_1 \wedge \cdots \wedge X(t)u_k = 0$ *if the linear system* (1.8) *is asymptotically stable.*

Let $|\cdot|$ denote a general vector norm in $\mathbf{R}^N$, $N = \binom{n}{k+r}$, and the matrix norm it induces for $N \times N$ matrices. Let μ be the corresponding Lozinskiĭ measure. Then the following corollary follows from the above theorem and Theorem A.3.4 of Appendix A.

Corollary 6.1.7. *Under the assumptions of Theorem* 6.1.6, $X(t)u_1 \wedge \cdots \wedge X(t)u_k \to 0$ *as* $t \to \infty$, *if*

$$\lim_{t \to \infty} \int_{t_0}^{t} \mu\left(A^{[k+r]}(s)\right) ds = -\infty. \tag{1.10}$$

In particular, if $\mu\left(A^{[k+r]}(s)\right) \leq -\delta < 0$, *then the convergence has exponential rate* δ.

§6.2. Nonlinear Autonomous Systems with an Invariant Linear Subspace

Let $D \subset \mathbf{R}^n$ be a convex open set, and $x \mapsto f(x) \in \mathbf{R}^n$ be a C^1 function defined in D. We consider the autonomous system in $\mathbf{R}^n$

$$x' = f(x) \tag{2.1}$$

under the following assumptions:

($\mathbf{H}_1$) The Jacobian matrix $\frac{\partial f}{\partial x}$ of the vector field f of (2.1) can be written as

$$\frac{\partial f}{\partial x}(x) = -\nu I + A(x) \qquad \text{for all } x \text{ in } D, \tag{2.2}$$

where ν is a constant, and $x \mapsto A(x)$ is a $n \times n$ matrix-valued function.

($\mathbf{H}_2$) There exists a constant matrix B with $\text{rank} B = r$, such that

$$BA(x) = 0 \qquad \text{for all } x \text{ in } D. \tag{2.3}$$

We will call a nonlinear system (2.1) satisfying ($\mathbf{H}_1$) and ($\mathbf{H}_2$) an autonomous system *with an invariant linear subspace.* Examples of such systems will be given later in the section.

The motivation for the name of such systems comes from the observation that the linear subspace $\ker B$ of $\mathbf{R}^n$ is invariant with respect to the linear variational equation

$$y'(t) = \frac{\partial f}{\partial x}\left(x(t, x_0)\right) y(t) \tag{2.4}$$

for any solution $x = x(t, x_0)$ of (2.1), in the sense that $y(t) \in \ker B$ for all $t \in \mathbf{R}$ if $y(0) \in \ker B$. This can be shown be observing that, after the change of variables

$y(t) = u(t)\exp(-\nu t)$, (2.4) becomes $u'(t) = A(x(t, x_0))u(t)$, satisfying the condition $BA(x(t, x_0)) = 0$ for all t. Therefore the claim follows from Theorem 6.1.1 in §6.1.

The following result shows that such a system has an invariant $(n-r)$-dimensional affine manifold.

Theorem 6.2.1. *Under the assumptions* $(\mathbf{H}_1)$ *and* $(\mathbf{H}_2)$, *the* $(n-r)$*-dimensional affine manifold*

$$\Gamma = \{x \in D \ : \ B(x - \bar{x}) = 0\}, \qquad \text{for some} \quad \bar{x} \in \mathbf{R}^n, \tag{2.5}$$

is invariant with respect to (1.1).

Proof. Since

$$\begin{aligned} B\,f(x) \ - \ B\,f(\bar{x}) \ &= \int_0^1 B\frac{\partial f}{\partial x}(\bar{x} + s(x - \bar{x}))\,ds\,(x - \bar{x}) \\ &= -\nu\,B\,(x - \bar{x}) \end{aligned}$$

for all x and $\bar{x}$ in $\overline{D}$, from (2.2). Choosing $\bar{x}$ as an equilibrium point of (2.1) in $\overline{D}$, we have

$$(Bx)' \ = \ -\nu B\,(x - \bar{x}). \tag{2.6}$$

It is easy to see that the invariance of Γ follows from (2.6). □

Remarks.

(i) As we shall see from the examples of systems with an invariant linear subspace given later in the section, possessing an invariant affine manifold is a main characteristic for such systems. However, as Theorem 6.2.1 demonstrates and, as we will discover throughout this section, the property of having an invariant linear subspace, namely satisfying ($\mathbf{H}_1$) and $(\mathbf{H}_2)$, determines much of the underlining geometry of the system (2.1).

(ii) We can see from (2.6) that if $\nu \neq 0$, then Γ is the global centre manifold in D (the stable centre manifold when $\nu > 0$, and the unstable centre manifold when

$\nu < 0$); if $\nu = 0$, Bx gives a set of r independent linear first integrals, and Γ is one of the level surfaces defined by these first integrals. In the former case, it is sufficient to study the flow generated by (2.1) on the global centre manifold Γ; even in the latter case, occasions may occur when we need to consider the flow on Γ. For instance, to study a specific periodic solution, it suffices to study the flow on the level surface Γ on which this periodic orbit lies.

(iii) Recall that if ϕ_t is the flow generated by (2.1) and $D\phi_t(x_0)$ is its tangent map at x_0, then $D\phi_t(x_0)u$ is a solution of the linear variational equation (2.4) for any $u \in \mathbf{R}^n$. Moreover, since Γ is an invariant manifold, if u is a vector tangent to Γ at $x_0 \in \Gamma$, then the vector $D\phi_t(x_0)u$ is tangent to Γ at $\phi_t(x_0)$. From this discussion and the fact that the tangent space of the affine manifold Γ is $\ker B$ at every point, we arrive at the following: (a) the subspace $\ker B$ of $\mathbf{R}^n$ is invariant with respect to (2.4); (b) we need only study those solutions of (2.4) which stay in $\ker B$ for all time if our interest is in the flow on Γ. This leads us to the same consideration as discussed in the remark following Theorem 6.1.1 in §6.1.

Before we discuss in detail the properties of the system (2.1) satisfying $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$, some concrete examples may help us to gain insight and intuition into such a system.

Examples of systems satisfying $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$.

(i). *The SEIRS models in Epidemiology.* These systems will be analyzed in full detail in Chapter VII, we thus refer readers to Chapter VII for their equations and properties.

(ii). *The chemostat models.* A typical chemostat model is given by the following

system of ordinary differential equations:

$$
\begin{aligned}
S' &= 1 - S - \frac{m_1 S x_1}{a_1 + S} - \frac{m_2 S x_2}{a_2 + S} \\
x_1' &= x_1 \Big(\frac{m_1 S}{a_1 + S} - 1 \Big) \\
x_2' &= x_2 \Big(\frac{m_2 S}{a_2 + S} - 1 \Big)
\end{aligned}
\tag{2.7}
$$

which models two competing micro-organisms x_1, x_2 feeding on the nutrient S in a chemostat, where $m_i, a_i,\ i = 1, 2$ are positive parameters. We can easily verify that (2.7) satisfies $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$ with $D = \mathbf{R}^3_+$, $\nu = 1$, $B = (1, 1, 1)$, $r = 1$, and that the 2-dimensional simplex in $\mathbf{R}^3$

$$\Gamma = \{ (S, x_1, x_2) \in \mathbf{R}^3_+ \ : \ S + x_1 + x_2 = 1 \}$$

is the affine invariant manifold. For detailed treatment on the chemostat models, readers are referred to [1] and [7], as well as the bibliographies in these papers.

(iii). *A chemostat model with an external inhibitor.* The following system is a variation of the chemostat model (2.7). In addition to the micro-organisms x_1, x_2 and the nutrient S, it has an external inhibitor p for the micro–organism x_1.

$$
\begin{aligned}
S' &= 1 - S - \frac{m_1 S x_1}{a_1 + S} f(p) - \frac{m_2 S x_2}{a_2 + S} \\
x_1' &= x_1 \Big(\frac{m_1 S}{a_1 + S} f(p) - 1 \Big) \\
x_2' &= x_2 \Big(\frac{m_2 S}{a_2 + S} - 1 \Big) \\
p' &= 1 - p - \frac{\delta x_2 p}{K + p}
\end{aligned}
\tag{2.8}
$$

Again $m_i, a_i,\ i = 1, 2$ and δ, K are positive parameters, and the function $f(p) \geq 0$ represents the degree of inhibition of p on the growth rate of x_1. Adding up all the equations of (2.8), we can see that the 3-dimensional simplex in $\mathbf{R}^4$

$$\Gamma = \{ (S, x_1, x_2, p) \in \mathbf{R}^4_+ \ : \ S + x_1 + x_2 = 1 \}$$

is the global centre manifold. Moreover, by calculating the Jacobian matrix of (2.8), we can verify that $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$ are satisfied, with $D = \mathbf{R}^4_+$, $\nu = 1$, $B = (1,1,1,0)$, and $r = 1$. Detailed treatment on (2.8) can be found in [7].

(iv). *A model in Chemical Kinetics.* The following system is considered in [14] as a model for the monomolecular reaction networks in chemical reactions:

$$\begin{aligned} x_1' &= -a\,x_1 + d\,x_3 \\ x_2' &= -(b+e)\,x_2 + a\,x_1 + c\,x_3 \\ x_3' &= (c+d)\,x_3 + b\,x_2 \\ x_4' &= e\,x_2 \end{aligned} \tag{2.9}$$

where $x_i \geq 0$ is the concentration of the reactant A_i, $i = 1, 2, 3, 4$, and a, b, c, d, e are reaction rates for various first order reactions involved. It is easy to check that (2.9) satisfies $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$ with $D = \mathbf{R}^4_+$, $v = 0$, $B = (1,1,1,1)$, $r = 1$, and the 3-dimensional hyperplane defined by

$$x_1 + x_2 + x_3 + x_4 = C$$

(which is called a conservation law), for each C, is a level surface for the linear first integral $x_1 + x_2 + x_3 + x_4$.

In the rest of the section, we readdress some of the nonlinear problems discussed in the earlier chapters in the context of autonomous systems satisfying $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$. Since these two conditions impose restrictions on the behaviour of the linear variational equations of (2.1), we would expect that these results hold for (2.1) under less restrictive conditions. In fact, as we will see throughout this section, the degree of relaxation on the conditions in these situations is directly related to $\text{rank}B$.

Our first subject is the upper estimation for the Hausdorff dimension of compact invariant sets for (2.1) under the assumptions $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$.

Let $K \subset \Gamma$ be a compact set invariant with respect to (2.1). Let

$$\lambda_1(x) \geq \lambda_2(x) \geq \cdots \geq \lambda_n(x)$$

be the eigenvalues of the symmetric matrix $\frac{1}{2}(\frac{\partial f}{\partial x}^* + \frac{\partial f}{\partial x})$. As a special case of Theorem 3.2.1 of Chapter III, we know that $\dim_H K < k + s$, for some integer $k \geq 0$ and real number $0 \leq s < 1$, provided

$$\lambda_1(x) + \cdots + \lambda_k(x) + s\,\lambda_{k+1}(x) \leq -\delta < 0 \tag{2.10}$$

for all $x \in K$.

The following result shows that the same estimate can be obtained for (2.1) under less restrictive conditions when $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$ are satisfied.

Theorem 6.2.2. *Assume that* (2.1) *satisfies* $(\mathbf{H}_1)$ *and* $(\mathbf{H}_2)$, *and that* $K \subset \Gamma$ *is a compact set invariant with respect to* (2.1). *Suppose, for some integer* $k \geq 0$ *and* $0 \leq s < 1$, $\delta > 0$,

$$\lambda_1(x) + \cdots + \lambda_{k+r}(x) + s\,\lambda_{k+r+1}(x) + r\,\nu \leq -\delta < 0 \tag{2.11}$$

for all $x \in K$. *Then* $\dim_H K < k + s$.

A remark is in order before we proceed to prove the theorem.

Remark. Since (2.10) implies $\lambda_{k+1}(x) < 0$, and thus $\lambda_{k+i}(x) < 0$, $i = 1, 2, \cdots, r+1$, we can see that the condition (2.11) is weaker than (2.10) in the case when $\nu = 0$. Moreover, the degree of relaxation (r) is equal to the number of independent linear first integrals (rank$B$).

Proof of Theorem 6.2.2. The proof is based on a remark made earlier in Chapter III, where we point out that $\dim_H K < k + s$ provided

$$\|\bigwedge^{k} D\phi_t(x)\|^{1-s}\, \|\bigwedge^{k+1} D\phi_t(x)\|^{s} \to 0 \quad \text{as} \quad t \to \infty$$

exponentially and the exponential rate of convergence is uniform with respect to all $x \in K$, where ϕ_t is the flow generated by (2.1), $D\phi_t(x)$ is the tangent map

of the diffeomorphism ϕ_t at x, and $\|\cdot\|$ denotes the euclidean norms in $\mathbf{R}^{\binom{n}{k}}$ and $\mathbf{R}^{\binom{n}{k+1}}$.

To illustrate the idea of the proof, we shall prove at first the case $\nu = 0$. The case $\nu \neq 0$ can be dealt with by the same idea and a change of variable.

Suppose $\nu = 0$. Let $x = x(t, x_0)$ be a solution to (2.1). The linear variational equation (2.4) with respect to $x(t, x_0)$ is a linear equation in $\mathbf{R}^n$ with a invariant subspace $V_0 = \ker B$ of $\mathbf{R}^n$. Therefore, from Proposition 6.1.5 of §6.1, its fundamental matrix $D\phi_t(x_0)$ satisfies the following:

$$
\begin{aligned}
&\|\bigwedge^{i} D\phi_t(x_0)(u_1 \wedge \cdots \wedge u_i)\| = \|D\phi_t(x_0)u_1 \wedge \cdots \wedge D\phi_t(x_0)u_i\| \\
&\leq \|D\phi_t(x_0)u_1 \wedge \cdots \wedge D\phi_t(x_0)u_i \wedge D\phi_t(x_0)w_1 \wedge \cdots \wedge D\phi_t(x_0)w_r\| \\
&= \|\bigwedge^{i+r} D\phi_t(x_0)(u_1 \wedge \cdots \wedge u_i \wedge w_1 \wedge \cdots \wedge w_r)\| \\
&\leq \|\bigwedge^{i+r} D\phi_t(x_0)\| \, \|u_1 \wedge \cdots \wedge u_i \wedge w_1 \wedge \cdots \wedge w_r\| \\
&\leq \|\bigwedge^{i+r} D\phi_t(x_0)\| \, \|u_1 \wedge \cdots \wedge u_i\|,
\end{aligned}
$$

which implies

$$\|\bigwedge^{i} D\phi_t(x_0)\| \leq \|\bigwedge^{i+r} D\phi_t(x_0)\|$$

for all $t > 0$ and integer $i \geq 1$, where $u_1, \cdots, u_i$ are linearly independent vectors in V_0 and $\{w_1, \cdots, w_r\}$ is an orthonormal basis for $V_0^{\perp}$.

Since $\bigwedge^{i+r} D\phi_t(x_0)$ is the fundamental matrix of the $(i+r)$-th compound equation of (2.4), we have, from Theorem A.3.1 of Appendix A

$$\|\bigwedge^{i+r} D\phi_t(x_0)\| \leq \exp \int_0^t [\lambda_1(x(\tau)) + \cdots + \lambda_{i+r}(x(\tau))] \, d\tau$$

for all $t > 0$. Therefore, letting $i = k$ and $k+1$, we have

$$\begin{aligned}
&\|\bigwedge^{k} D\phi_t(x_0)\|^{1-s} \, \|\bigwedge^{k+1} D\phi_t(x_0)\|^{s} \\
&\le \exp \int_0^t \left[(1-s)\big(\lambda_1 + \cdots + \lambda_{k+r}\big) + s\big(\lambda_1 + \cdots + \lambda_{k+r+1}\big)\right] d\tau \\
&= \exp \int_0^t \big(\lambda_1 + \cdots + \lambda_{k+r} + s\lambda_{k+r+1}\big)\, d\tau
\end{aligned}$$

which shows that $\|\bigwedge^{k} D\phi_t(x_0)\|^{1-s}\, \|\bigwedge^{k+1} D\phi_t(x_0)\|^{s}$ has the convergence property we require. Therefore $\dim_H K < k+1$ in this case.

Now for the general ν, let $u(t) = y(t)\exp(\nu t)$, then the linear variational equation (2.4) becomes

$$u'(t) = A(x(t, x_0))\, u(t)$$

satisfying $BA(x(t, x_0)) = 0$ for all $t > 0$ and all $x_0 \in D$. From the proof for the case $\nu = 0$, we have

$$\|\bigwedge^{i} (D\phi_t \exp(\nu t))\| \le \|\bigwedge^{i+r} (D\phi_t \exp(\nu t))\|$$

Thus

$$\begin{aligned}
\|\bigwedge^{i} D\phi_t\| &\le \|\bigwedge^{i+r} D\phi_t\| \exp(r\nu t) \\
&\le \exp \int_0^t (\lambda_1 + \cdots + \lambda_{i+r} + r\nu)\, d\tau
\end{aligned}$$

for $i = k$ and $k+1$. Now it is easy to see that (2.11) implies the same convergence property required for $\|\bigwedge^{k} D\phi_t\|^{1-s} \|\bigwedge^{k+1} D\phi_t\|^{s}$. Therefore the theorem is proved.

We would like to note that the sum $\lambda_1 + \cdots + \lambda_k$ of the k largest eigenvalues of $\frac{1}{2}\big(\frac{\partial f}{\partial x}^{*} + \frac{\partial f}{\partial x}\big)$ is the Lozinskiĭ measure of the k-th compound matrix $\frac{\partial f}{\partial x}^{[k]}$ of $\frac{\partial f}{\partial x}$ corresponding to the euclidean norm on $\mathbf{R}^{\binom{n}{k}}$. If a general vector norm $|\cdot|$ and the corresponding Lozinskiĭ measure are used in the above proof instead, Theorem 2.2.1 of Chapter II can give us the following more general and more flexible

form of the above Theorem 6.2.2, which reduces to Theorem 6.2.2 when the norms are euclidean.

Theorem 6.2.3. *Under the assumptions of Theorem* 6.2.2, *if, for some Lozinskiĭ measure* μ,

$$(1-s)\,\mu\left(\frac{\partial f}{\partial x}^{[k+r]}\right) + s\,\mu\left(\frac{\partial f}{\partial x}^{[k+r+1]}\right) < -r\nu \tag{2.12}$$

on K, *then* $\dim_H K < k+s$.

Remark. Theorems 3.2.4, 3.3.2, 3.3.3, and 3.3.6 in Chapter III can also be generalized to (2.1) satisfying $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$ in the spirit of Theorems 6.2.1 and 6.2.2.

Our next topic is to derive Bendixson and Dulac criteria for autonomous systems (2.1) satisfying $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$. The interest is in the conditions which preclude existence of periodic orbits on the invariant manifold Γ.

Suppose Γ is simply connected. Let B be the unit ball in $\mathbf{R}^2$ with boundary ∂B and closure $\overline{B}$. For a simple closed rectifiable curve $\psi \in \mathrm{Lip}(\partial B \to \Gamma)$, the set

$$\Sigma(\psi, \Gamma) = \{\varphi \in \mathrm{Lip}(\overline{B} \to \Gamma) \;:\; \varphi(\partial B) = \psi(\partial B)\}$$

is not empty (see Chapter IV, §4.2.1). Consider the functional $\mathcal{A}$ on $\mathrm{Lip}(\overline{B} \to \Gamma)$ defined by

$$\mathcal{A}\varphi = \int_B \left\| \frac{\partial \varphi}{\partial p_1} \wedge \frac{\partial \varphi}{\partial p_2} \right\| \tag{2.13}$$

where $p = (p_1, p_2) \in B$, and the norm $\|\cdot\|$ is the euclidean norm on $\mathbf{R}^N$, $N = \binom{n-r}{2}$. Therefore $\mathcal{A}\varphi$ is the usual surface area of $\varphi(\overline{B})$. The following result follows from Proposition 4.2.2 in Chapter IV.

Proposition 6.2.4. *Suppose* ψ *is a simple closed rectifiable curve in* Γ. *Then there exists a* $\delta > 0$ *such that*

$$\mathcal{A}\varphi \geq \delta$$

for all $\varphi \in \Sigma(\psi, \Gamma)$.

Let μ be the Lozinskiĭ measure corresponding to a vector norm on $\mathbf{R}^{\binom{n}{r+2}}$. The following result shows that the condition

$$\mu\left(\frac{\partial f^{[r+2]}}{\partial x}\right) < -r\nu \tag{2.14}$$

is a Bendixson criterion for (2.1) satisfying $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$.

Theorem 6.2.5. *Assume that* (2.1) *satisfies* $(\mathbf{H}_1)$ *and* $(\mathbf{H}_2)$ *and*

(*a*) *either* Γ *is bounded or* Γ *contains a bounded absorbing set,*

(*b*) (2.14) *holds on* Γ.

Then no simple closed rectifiable curve in Γ *can be invariant with respect to* (2.1).

Proof. As we have seen in the proof of Theorem 6.2.2, it suffices to prove the theorem for the case $\nu = 0$. The general case can be dealt with in the same way after a change of variables in the linear variational equation (2.4).

For each $x_0 \in \Gamma$, the solution $x = x(t, x_0)$ to (2.1) exists and stays in Γ for all $t > 0$. The linear variational equation of (2.1) with respect to $x(t, x_0)$ can now be written as

$$y'(t) = A(x(t, x_0))\, y(t) \tag{2.15}$$

which satisfies $BA(x(t, x_0)) = 0$ for all $t > 0$. It then follows from Theorem 6.1.6 in the first section of this chapter that (2.14) implies $y_1(t) \wedge y_2(t) \to 0$ as $t \to \infty$, for any two solutions $y_1(t), y_2(t)$ of (2.15) such that $By_i(0) = 0$, $i = 1, 2$, uniformly with respect to all $x_0 \in \Gamma$.

Suppose now $\psi \in \mathrm{Lip}(\partial B \to \Gamma)$ is a simple closed rectifiable curve in Γ which is invariant with respect to (2.1), and $\varphi \in \Sigma(\psi, \Gamma)$. Let $\varphi_t(p) = x(t, \varphi(p))$, $p = (p_1, p_2) \in \overline{B}$. Then $\varphi_t \in \Sigma(\psi, \Gamma)$ for each $t > 0$ by the invariance of ψ. Therefore $\mathcal{A}\varphi_t \geq \delta > 0$ for all $t > 0$. Moreover,

$$\frac{\partial \varphi_t}{\partial p_i} = \frac{\partial x(t, \varphi)}{\partial x_0} \frac{\partial \varphi}{\partial p_i} \qquad i = 1, 2,$$

are two linearly independent solutions to (2.15) with respect to $\varphi_t(p)$, and $B\frac{\partial \varphi_0}{\partial p_i} = B\frac{\partial \varphi}{\partial p_i} = 0$, $i = 1, 2$. Hence

$$\frac{\partial \varphi_t}{\partial p_1} \wedge \frac{\partial \varphi_t}{\partial p_2} \to 0 \qquad \text{as} \quad t \to \infty$$

uniformly for almost all $p \in \overline{B}$. Therefore $\mathcal{A}\varphi_t \to 0$ as $t \to \infty$. This contradicts $\mathcal{A}\varphi_t \geq \delta$. The theorem is proved. □

Some remarks are in order.

Remarks.

(i) We would like to note that in the proof of Theorem 6.2.5 only the flow of (2.1) restricted to the invariant affine manifold Γ is relevant. Therefore, the theorem still holds if we relax the assumptions $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$ to the following:

$(\mathbf{H}_1')$ There exists a constant matrix B such that $\operatorname{rank} B = r$ and the affine manifold

$$\Gamma = \{x \in D \ : \ B(x - \bar{x}) = 0\} \quad \text{for some} \ \ \bar{x} \in D$$

is invariant with respect to (2.1).

$(\mathbf{H}_2')$ The Jacobian matrix $\frac{\partial f}{\partial x}$ of f can be written as

$$\frac{\partial f}{\partial x}(x) \ = \ -\nu\, I_{n\times n} \ + \ A(x) \qquad \text{on} \quad \Gamma,$$

and

$$BA(x) \ = \ 0 \qquad \text{on} \quad \Gamma.$$

(ii) Theorems 5.1.3, 5.1.4, 5.1.7 can also be modified for (2.1) satisfying $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$.

For example, suppose Γ is bounded or contains a bounded absorbing set. Let $x \mapsto V(x, y)$ be a real-valued function defined on $\Gamma \times \mathbf{R}^N$, $N = \binom{n}{r+2}$. Define $\overset{\bullet}{V}$ by

$$\overset{\bullet}{V}(x,y) = \lim_{h \to 0+} \frac{1}{h} \left[V\left(x + hf(x), \ y + h(\frac{\partial f}{\partial x}^{[r+2]} + r\nu I)y \right) - V(x,y) \right]$$

Then

$$\overset{\bullet}{V} = \frac{\partial V^*}{\partial x} f(x) + \frac{\partial V^*}{\partial y} \left(\frac{\partial f^{[r+2]}}{\partial x} + r\nu I \right) y$$

almost everywhere if V is assumed to be locally Lipschitz. Now the conclusion of Theorem 6.2.5 holds if V satisfies the following conditions:

$$V(x,y) \geq a|y|, \quad \text{and} \quad \overset{\bullet}{V}(x,y) \leq -b|y|, \tag{2.16}$$

for all $(x,y) \in \Gamma \times \mathbf{R}^N$, where $|\cdot|$ is a vector norm in $\mathbf{R}^N$, and $a, b > 0$ are some constants.

Let $x \mapsto A(x)$ be a $N \times N$ matrix-valued function which is C^1 and non-singular for $x \in \Gamma$. Define $V(x,y) = |A(x)y|$. Then V satisfies (2.16) if the following general Dulac condition

$$\mu\big(A_f A^{-1} + A \frac{\partial f^{[r+2]}}{\partial x} A^{-1}\big) < -r\nu \tag{2.17}$$

holds on Γ.

Suppose $D_0 \subset \bar{D}_0 \subset \Gamma$ is a bounded absorbing set in Γ. Then no simple closed rectifiable curve in Γ can be invariant with respect to (2.1) if $\bar{q}_{r+2}(D_0) < -r\nu$, where $\bar{q}_{r+2}(D_0)$ is defined as

$$\bar{q}_{r+2}(D_0) = \limsup_{t \to \infty} \bar{q}_{r+2}(t, D_0) \tag{2.18}$$

and

$$\bar{q}_{r+2}(t, D_0) = \sup_{x \in D_0} \frac{1}{t} \int_0^t \mu\big(B(x(s,x_0))\big)\, ds \tag{2.19}$$

with

$$B = A_f A^{-1} + A \frac{\partial f^{[r+2]}}{\partial x} A^{-1}. \tag{2.20}$$

(iii) Once again, these conditions are directly related to $r = \text{rank}B$. In particular, when $\nu = 0$, they are determined by the number of independent linear first integrals.

(iv) When the Lozinskiĭ measure μ is calculated corresponding to the l_1, l_∞, l_2 norms on $\mathbf{R}^N$, $N = \binom{n}{r+2}$, (2.15) gives rise to following concrete conditions each is Bendixson's criterion when $n = 2$ and $r = 0$.

$$\sup_{(i)}\left\{\frac{\partial f_{i_1}}{\partial x_{i_1}} + \cdots + \frac{\partial f_{i_{r+2}}}{\partial x_{i_{r+2}}} + \sum_{k \neq i_1, \cdots, i_{r+2}} \left(\left|\frac{\partial f_{i_1}}{\partial x_k}\right| + \cdots + \left|\frac{\partial f_{i_{r+2}}}{\partial x_k}\right|\right)\right\} < -r\nu,$$

$$\sup_{(i)}\left\{\frac{\partial f_{i_1}}{\partial x_{i_1}} + \cdots + \frac{\partial f_{i_{r+2}}}{\partial x_{i_{r+2}}} + \sum_{k \neq i_1, \cdots, i_{r+2}} \left(\left|\frac{\partial f_k}{\partial x_{i_1}}\right| + \cdots + \left|\frac{\partial f_k}{\partial x_{i_{r+2}}}\right|\right)\right\} < -r\nu,$$

$$\lambda_1 + \cdots + \lambda_{r+2} < -r\nu,$$

respectively.

(v) When $r = n-2, r+2 = n$, and $N = \binom{n}{r+2} = 1$. Therefore $A(x)$ in (2.17) is a real-valued function $\alpha(x)$, and (2.17) becomes

$$\operatorname{div}(\alpha f) < -(n-2)\nu.$$

For example, when $n = 3$, and $\nu = 0$. $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$ imply that (2.1) has a linear first integral. Our results here say (2.1) has no periodic orbits if the Dulac condition $\operatorname{div}(\alpha f) < 0$ is satisfied on Γ. This is also in accordance with the fact that the flow of (2.1) on Γ is 2-dimensional.

(vi) As we have demonstrated in Chapters IV and V, our Bendixson criterion and Dulac criteria preclude the existence in Γ of orbits of the following types:

(1) periodic orbits;

(2) homoclinic orbits;

(3) a pair of heteroclinic orbits of the same equilibria;

(4) heteroclinic cycles.

Our next result concerns the orbital stability of periodic orbits to (2.1) satisfying $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$.

Let $x = p(t)$ be a periodic solution to (2.1) of least period ω. Assume that its orbit $\gamma = \{p(t) \ : \ 0 \leq t < \omega\}$ is contained in the invariant manifold Γ.

Theorem 6.2.6. *Assume that* (2.1) *satisfies* $(\mathbf{H}_1)$ *and* $(\mathbf{H}_2)$. *Then a sufficient condition for a periodic orbit* $\gamma = \{p(t) \;:\; 0 \le t < \omega\} \subset \Gamma$ *to be asymptotically orbitally stable with asymptotic phase is that the linear system*

$$z'(t) = \left(\frac{\partial f^{[r+2]}}{\partial x}(p(t)) + r\nu I \right) z(t) \tag{2.21}$$

is asymptotically stable.

Using the relation of Lozinskiĭ measure with the stability of linear systems given in Theorem 4.2 of Appendix A, we have the following corollary.

Corollary 6.2.7. *Under the assumptions of Theorem* 6.2.6 , γ *is asymptotically orbitally stable with asymptotic phase if*

$$\int_0^\omega \mu \left(\frac{\partial f^{[r+2]}}{\partial x}(p(t)) \right) dt < -r\,\nu \tag{2.22}$$

where μ *is the Lozinskiĭ measure corresponding to a vector norm in* $\mathbf{R}^N$, $N = \binom{n}{r+2}$.

Proof of Theorem 6.2.6. It suffices to prove the theorem assuming $\nu = 0$. The general case can be proved in the same way after a change of variables in the linear variational equation of (2.1). From $(\mathbf{H}_1)$, the linear variational equation (2.4) with respect to $p(t)$ can be written as

$$y'(t) = A(p(t))\, y(t). \tag{2.23}$$

satisfying $BA(p(t)) = 0$ for all $t > 0$. From the Floquet theory (see [5]), a fundamental matrix $Y(t)$ of (2.23) can be written in the form

$$Y(t) = P(t)\exp(Lt).$$

From Theorem 6.1.1 in §6.1, we know $BY(t) = B$ for all t. In particular, $B\exp(L\omega) = B$, which implies

$$\exp(L^*\omega)B^* = B^*.$$

Since rank $B = r$, the matrix L^*, hence L, has at least $r+1$ eigenvalues equal to 0, counting the one correspond to the nonconstant periodic solution $x = p(t)$. Now the real parts of the characteristic exponents of the system (2.21) are all negative if (2.21) is asymptotically stable. Using the same argument as in the proof of Theorem 4.4.2 in Chapter IV, we know that these characteristic exponents are the eigenvalues of $L^{[r+2]}$ which are of the form

$$\lambda_{i_1} + \cdots + \lambda_{i_{r+2}}, \qquad 1 \leq i_1 < \cdots < i_k \leq n,$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of L. Since L has at least $(r+1)$ eigenvalues zero, it follows that the remaining $(n-r-1)$ eigenvalues of L are also characteristic exponents of (2.21), thus have negative real parts. Hence, γ is asymptotically orbitally stable with asymptotic phase. $\square$

We have seen in Chapter V that, for a general autonomous system, a Bendixson Criterion which is robust under C^1 local perturbations of the vector field at all nonequilibrium nonwandering points will imply that every nonwandering point is an equilibrium. The proof relies on Pugh's C^1 Closing Lemma for vector fields. To develop similar results for autonomous systems having an invariant linear subspace, it is essential to establish a C^1 Closing Lemma for such systems. More precisely, let $f \in C^1(D \to \mathbf{R}^n)$ be a vector field satisfying $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$ with a constant matrix B, and an invariant manifold Γ determined by B. Suppose that $x_0 \in \Gamma$ is a nonwandering point for (2.1) and that $f(x_0) \neq 0$. We want to investigate the possibility of closing up the nonwandering orbit at x_0 by smoothly perturbing f locally near x_0 so that the resulting vector field g also satisfies $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$. Moreover, the same affine manifold Γ is also invariant with respect to the corresponding differential equation

$$x' = g(x) \tag{2.24}$$

This is equivalent to requiring that the perturbation g satisfy $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$ and share the same constant matrix B with f. Considering the remark (i) follow-

ing the proof of Theorem 6.2.5, we need only require that g satisfies the weaker assumptions $(\mathbf{H}_1^{'})$ and $(\mathbf{H}_2^{'})$. The following result is an adaptation of Pugh's Closing Lemma to systems satisfying $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$, a proof of which is provided in the Appendix D.

Lemma 6.2.8. *Let $f \in C^1(D \to \mathbf{R}^n)$ be a vector field satisfying $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$. Suppose that $x_0 \in \Gamma$ is a nonwandering point for (2.1) and that $f(x_0) \neq 0$. Then for each neighbourhood U of x_0 and $\epsilon > 0$, there exists a C^1 ϵ-perturbation of f, $g \in C^1(D \to \mathbf{R}^n)$, such that*

(1) $\operatorname{supp}(f - g) \subset U$,

(2) *g satisfies $(\mathbf{H}_1^{'})$ and $(\mathbf{H}_2^{'})$ with the same matrix B,*

(3) *the system (2.20) has a nonconstant periodic solution whose trajectory intersects U.*

Remark. We want to emphasize again that g is only required to satisfy the weaker assumptions $(\mathbf{H}_1^{'})$ and $(\mathbf{H}_2^{'})$.

Once this is established, we can derive autonomous convergence theorems for systems (2.1) satisfying $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$, based on the Bendixson Criterion (2.14) obtained in Theorem 6.2.5 earlier in this section.

Theorem 6.2.9. *Assume that Γ is simply connected. Suppose*

(*a*) *Γ is either bounded or contains a bounded absorbing set,*

(*b*) *(2.14) is satisfied on Γ.*

Then:

(*c*) *Every nonwandering point of D is an equilibrium.*

(*d*) *Every nonempty alpha or omega limit set in D is a single equilibrium.*

(*e*) *Any equilibrium in D is the alpha limit set of at most two distinct nonequilibrium trajectories.*

Now suppose $\bar{x}$ is the only equilibrium in Γ. Then Theorem 6.2.9 gives the

following result on the global stability of $\bar{x}$. The proof is parallel to that of Theorem 5.3.6 in Chapter V.

Theorem 6.2.10. *Suppose that*

(*a*) (2.1) *satisfies* $(\mathbf{H}_1)$ *and* $(\mathbf{H}_2)$,

(*b*) Γ *is either bounded or contains a bounded absorbing set,*

(*c*) (2.14) *is satisfied on* Γ.

(*d*) $\bar{x}$ *is the only equilibrium in* Γ.

Then $\bar{x}$ *is globally asymptotically stable in* Γ.

Remarks.

(i) When $\nu \neq 0$, Γ is globally attracting. Therefore, in Theorems 6.2.9, 6.2.10, Γ can be replaced by D.

(ii) Obviously, condition (2.14) can be replace by more general Dulac type conditions discussed in the remark (ii) following the proof of Theorem 6.2.5.

§6.3. Autonomous Systems with an Invariant Affine Manifold

The method developed in §6.2 can be used to derive similar results for a more general class of autonomous systems. In this section, we discuss the class of autonomous systems of differential equations possessing many invariant affine manifolds. Before we give the formal definition, we would like to present an example which motivates our consideration.

The following system of equations are proposed by M. Eigen and P. Schuster [4] (see also [6]) as a model for the self-organizing and self-regulating phenomena observed in prebiotic evolutions and in animal behaviour,

$$x_i' = x_i \left[G_i(x_1, \cdots, x_n) - \phi \right] \qquad i = 1, \cdots, n, \tag{3.1}$$

where $\phi = \sum_{i=1}^n x_i G_i$, and all the variables x_i are assumed to be nonnegative.

Eigen and Schuster call such a system *a dynamical system under constant organization.* In the special case when each G_i is linear, it is called *hypercycle.* One prominent feature of such a system is that it leaves the boundary and all faces of the following region

$$D = \{x \in \mathbf{R}_+^n \ : \ \sum_{i=1}^n x_i \leq 1\}$$

invariant.

Observe that adding all the equations in (3.1) leads to the following

$$(\sum_{i=1}^n x_i)' = \phi(1 - \sum_{i=1}^n x_i). \tag{3.2}$$

This implies the $(n-1)$-dimensional simplex

$$\Gamma = \{x \in \mathbf{R}_+^n \ : \ \sum_{i=1}^n x_i = 1\} \tag{3.3}$$

is invariant with respect to (3.1).

However, identity (3.2) is different from the corresponding identity (2.6) for systems satisfying $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$ in that ϕ is a nonconstant function.

If we look at the Jacobian matrix of (3.1), we will see further differences between (3.1) and systems satisfying $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$. Let $f(x) = (f_1(x), \cdots, f_n(x))$ denote the vector field of (3.1). Then $f_i(x) = x_i(G_i - \phi)$, for each i. Summing up elements in each column of the Jacobian matrix $\frac{\partial f}{\partial x}$, we have

$$\sum_{i=1}^n \frac{\partial f_i}{\partial x_j} = -\phi + \frac{\partial \phi}{\partial x_j}\left(1 - \sum_{i=1}^n x_i\right), \qquad \text{for each} \quad j.$$

This time, the sum is the same ($-\phi$) for each column only on the invariant manifold Γ.

This example motivates the following hypothesis for this class of systems.

$(\mathbf{H}_3)$ There is a constant matrix B such that $\operatorname{rank} B = r$ and the $(n-r)$-dimensional affine manifold

$$\Gamma = \{x \in D \ : \ B(x - \bar{x}) = 0\} \qquad \text{for some } \bar{x} \in D \tag{3.4}$$

is invariant with respect to (2.1);

(H$_4$) The Jacobian matrix $\frac{\partial f}{\partial x}$ can be written as

$$\frac{\partial f}{\partial x}(x) = -\nu(x) I_{n\times n} + A(x) \qquad \text{on} \quad \Gamma, \tag{3.5}$$

and

$$B\,A(x) = 0 \qquad \text{on} \quad \Gamma, \tag{3.6}$$

where $x \mapsto \nu(x)$ is a real-valued function.

The autonomous system (2.1) satisfying $(\mathbf{H}_3)$ and $(\mathbf{H}_4)$ will be called an autonomous system *with an invariant affine manifold.* Note that we do not require $\nu(x)$ to be of one sign in D. We write $-\nu(x)$ in (3.5) for easy comparison with the corresponding condition in $(\mathbf{H}_1)$.

For the model (3.1), the affine manifold is the simplex in (3.3), B is the $1 \times n$ matrix $(1, \cdots, 1)$, $r = 1$, and $\nu(x) = \phi(x)$.

We shall see in the following that, for such a system, the subspace $\ker B$ of $\mathbf{R}^n$ is only invariant under the linear variational equations (2.4) for those solutions staying in the invariant manifold Γ. Let $x_0 \in \Gamma$; then the solution $x(t, x_0)$ to (2.1) satisfies $x(t, x_0) \in \Gamma$ for each t at which it exists. Therefore, from (3.5), the linear variational equation (2.4) for $x(t, x_0)$ can be written as

$$y'(t) = -\nu(x(t, x_0))\, y(t) + A(x(t, x_0))\, y(t). \tag{3.7}$$

The change of variables $u(t) = y(t) \exp \int_0^t \nu(x(s, x_0)) ds$ leads to

$$u'(t) = A(x(t, x_0))\, u(t) \tag{3.8}$$

satisfying $BA(x(t, x_0)) = 0$ for all t. Therefore (3.8) is a linear system having an invariant subspace $\ker B$ discussed in §6.1. As a consequence, $\ker B$ is also invariant with respect to (3.7).

Now it is not hard to see that, once we restrict our attention to the invariant affine manifold Γ, results obtained for autonomous systems (2.1) satisfying $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$ can be proved in exactly the same way for (2.1) which satisfies $(\mathbf{H}_3)$

and $(\mathbf{H}_4)$, only replacing the constant ν by the function $\nu(x)$. We will simply state the corresponding results without giving detailed proof.

Theorem 6.3.1. *Assume that* (2.1) *satisfies* $(\mathbf{H}_3)$ *and* $(\mathbf{H}_4)$, *and that the compact set* $K \subset \Gamma$ *is invariant with respect to* (2.1) . *Suppose, for some integer* $k \geq 0$ *and* $0 \leq s < 1$,

$$(1-s)\,\mu\left(\frac{\partial f}{\partial x}^{[k+r]}(x)\right) + s\,\mu\left(\frac{\partial f}{\partial x}^{[k+r+1]}(x)\right) < -r\nu(x) \tag{3.9}$$

on K. *Then* $\dim_H K < k+s$.

Theorem 6.3.2. *Assume that* (2.1) *satisfies* $(\mathbf{H}_3)$ *and* $(\mathbf{H}_4)$ *and*

(*a*) Γ *is simply connected,*

(*b*) *either* Γ *is bounded or* Γ *contains a bounded absorbing set,*

(*c*) $\mu\left(\frac{\partial f}{\partial x}^{[r+2]}(x)\right) < -r\,\nu(x)$ *on* Γ.

Then no simple closed rectifiable curve in Γ *can be invariant with respect to* (2.1).

Let $x = p(t)$ be a periodic solution to (2.1) of least period ω. Assume that its orbit $\gamma = \{p(t) \;:\; 0 \leq t < \omega\}$ is contained in the invariant manifold Γ. Modifying Theorem 6.2.6 and Corollary 6.2.7, we have the following results on the orbital stability of γ when $(\mathbf{H}_3)$ and $(\mathbf{H}_4)$ are assumed.

Theorem 6.3.3. *Assume that* (2.1) *satisfies* $(\mathbf{H}_3)$ *and* $(\mathbf{H}_4)$. *Then a sufficient condition for a periodic orbit* $\gamma = \{p(t) \;:\; 0 \leq t < \omega\} \subset \Gamma$ *be asymptotically orbitally stable with asymptotic phase is that the linear system*

$$z'(t) = \left(\frac{\partial f}{\partial x}^{[r+2]}(p(t)) + r\nu\big(p(t)\big)I\right) z(t) \tag{3.10}$$

is asymptotically stable.

Using the relation of Lozinskiĭ measure with the stability of linear systems given in Theorem 4.2 of Appendix A, we have the following corollary.

Corollary 6.3.4. *Under the assumptions of Theorem* 6.3.3, γ *is asymptotically orbitally stable with asymptotic phase if*

$$\int_0^{\omega} \mu\left(\frac{\partial f}{\partial x}^{[r+2]}(p(t)) + r\,\nu(p(t))\right) dt < 0 \tag{3.11}$$

where μ *is the Lozinskiĭ measure corresponding to a vector norm in* $\mathbf{R}^N$, $N = \binom{n}{r+2}$.

Remark. Theorems 6.3.2, 6.3.3 remain valid under more general Dulac type conditions discussed in the remark (i) following Theorem 6.2.5 in §6.2.

Given in the following is another example of systems satisfying $(\mathbf{H}_3)$ and $(\mathbf{H}_4)$ but not $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$.

Example. *A homogeneous system of degree* 1. Consider a autonomous system of the form

$$x' = Ax + f(x) \qquad x \in \mathbf{R}^n_+, \tag{3.12}$$

where $A = (a_{ij})_{n\times n}$ is a $n \times n$ constant matrix, and $x \mapsto f(x)$ is a continuous function and is homogeneous of degree 1, namely $f(\lambda x) = \lambda f(x)$ for all $\lambda > 0$ and $x \in \mathbf{R}^n$. The components x_i of x are restricted to be nonnegative, and we assume the components f_i of f satisfy $f_i|_{x_i=0} \geq 0$. Hence the positive cone $\mathbf{R}^n_+$ of $\mathbf{R}^n$ is invariant with respect to (3.12).

The system (3.12) can be transformed into a system satisfying $(\mathbf{H}_3)$ and $(\mathbf{H}_4)$. Introducing the variable

$$y = \frac{x}{\sum_{i=1}^n x_i},$$

we note that $\sum_{i=1}^n y_i = 1$, and y satisfies

$$y' = Ay - \left(\sum_{i,j=1}^n a_{ij}y_j\right) y - \left(\sum_{i=1}^n f_i(y)\right) y + f(y). \tag{3.13}$$

Adding all the equations in (3.13) we have

$$\left(\sum_{i=1}^n y_i\right)' = \left(\sum_{i,j=1}^n a_{ij}y_j + \sum_{i=1}^n f_i(y)\right)\left(1 - \sum_{i=1}^n y_i\right).$$

Therefore the $(n-1)$ simplex

$$\Gamma = \left\{ y \in \mathbf{R}_+^n \ : \ \sum_{i=1}^n y_i = 1 \right\} \tag{3.14}$$

is invariant with respect to (3.12).

Calculating the Jacobian matrix J of the vector field of (3.12), we have

$$J = A - \left(\sum_{i,j=1}^n a_{ij} y_j + \sum_{i=1}^n f_i(y) \right) I + \frac{\partial f}{\partial y}$$

$$- y\Big(\sum_{i=1}^n a_{i_1}, \cdots, \sum_{i=1}^n a_{i_n}\Big) - y\Big(\sum_{i=1}^n \frac{\partial f_i}{\partial y_1}, \cdots, \sum_{i=1}^n \frac{\partial f_i}{\partial y_n}\Big).$$

Adding the elements in each column of J, we have, for the j-th one,

$$-\left(\sum_{i,j=1}^n a_{ij} y_j + \sum_{i=1}^n f_i(y) \right) + \left(1 - \sum_{i=1}^n y_i \right) \left(\sum_{i=1}^n a_{ij} + \sum_{i=1}^n \frac{\partial f_i}{\partial y_j} \right).$$

Therefore, on Γ, the sum is the same for each column and is equal to

$$-\left(\sum_{i,j=1}^n a_{ij} y_j + \sum_{i=1}^n f_i(y) \right). \tag{3.15}$$

This shows that (3.12) satisfies $(\mathbf{H}_3)$ and $(\mathbf{H}_4)$ with $D = \mathbf{R}_+^n$, $B = (1, \cdots, 1)$, $r = 1$, Γ given in (3.14) and $-\nu(x)$ given in (3.15).

§6.4. Autonomous Systems with First Integrals

In this section, we will develop similar results obtained in sections §6.2, §6.3, particularly the Bendixson and Dulac criteria, for autonomous systems having nonlinear first integrals. In systems which model physical phenomena, existence of first integrals corresponds to the presence of certain conservation laws, such as conservation of mass, energy, momentum, etc. Efforts will be made to demonstrate the similarities between this nonlinear theory and the linear one in previous sections. We will explore the restriction of the existing conservation laws on the linear variational equations, and the implications of this to the geometry of the flow. The exterior product technique presented in section §6.1 again plays an important role.

Let $D \subset \mathbf{R}^n$ be an open subset, and $x \mapsto f(x) \in \mathbf{R}^n$ be a function which is C^1 for $x \in D$. We consider the autonomous system in $\mathbf{R}^n$

$$x' = f(x). \tag{4.1}$$

A real-valued function $x \mapsto H(x)$ defined for $x \in D$ and not identically constant in D is said to be a *first integral* for (4.1) if $H(x(t, x_0))$ remains constant along each solution $x(t, x_0)$ of (4.1). If H is C^1, this is equivalent to H satisfying the following condition

$$\frac{\partial H}{\partial x}^* f(x) = 0 \qquad \text{for all} \quad x \in D. \tag{4.2}$$

Throughout this section, we will only deal with first integrals which are C^1. Suppose $H(x)$ is a first integral for (4.1). Then, for each $c > 0$, the surface $H(x) = c$ in $\mathbf{R}^n$ will be called a *level surface* determined by H. It follows from (4.2) that each trajectory necessarily stays on one of the level surfaces. In particular, each periodic trajectory stays on a level surface. We denote by $\nabla H(x)$ the gradient vector of H. All vectors in $\mathbf{R}^n$ will be assumed as column vectors. We also use $\langle , \rangle$ and $\|\cdot\|$ to denote the euclidean inner product and norm in $\mathbf{R}^n$, respectively.

Since $H(x(t, x_0)) = H(x_0)$ for all t and $x_0 \in D$, we have, by differentiating with respect to x,

$$\nabla H(x_0) = \frac{\partial x}{\partial x_0}^*(t, x_0) \nabla H\big(x(t, x_0)\big)$$

where the asterisk denotes the transposition of matrices. Therefore for each $u \in \mathbf{R}^n$, we have

$$\begin{aligned} \langle \nabla H(x_0), u \rangle &= \langle \frac{\partial x}{\partial x_0}^*(t, x_0) \nabla H\big(x(t, x_0)\big), u \rangle \\ &= \langle \nabla H\big(x(t, x_0)\big), \frac{\partial x}{\partial x_0}(t, x_0) u \rangle \end{aligned}$$

for all t and all $x_0 \in D$. Note that $y(t) = \frac{\partial x}{\partial x_0}(t, x_0)u$ is a solution to the linear variational equation

$$y'(t) = \frac{\partial f}{\partial x}\big(x(t, x_0)\big)\, y(t). \tag{4.3}$$

We arrive at the following result.

Proposition 6.4.1. *Let* H *be a first integral for* (4.1), *and* $y = y(t)$ *be a solution to the variational equation* (4.3). *Then*

$$\langle \nabla H\big(x(t,x_0)\big),\, y(t)\rangle = \langle \nabla H(x_0),\, y(0)\rangle \tag{4.4}$$

for all $x_0 \in D$, *and* $t \in \mathbf{R}^n$.

Remarks.

(i) This result is in the same spirit as Proposition 6.1.3 for the affine case.

(ii) Since $\langle \nabla H\big(x(t,x_0)\big), y(t)\rangle$ represents the length of the projection of $y(t)$ onto the vector $\nabla H\big(x(t,x_0)\big)$, (4.4) may be interpreted geometrically as that the linearization of the flow for (4.1) preserves the infinitesimal length in the normal direction of each level surface.

(iii) We will demonstrate in the following that the geometrical observation given in Proposition 6.4.1 is the foundation for all development in this section, just as Proposition 6.1.3 is for the affine case.

(iv) As a special case of (4.4), letting $y(t) = \frac{\partial x}{\partial x_0}(t,x_0)\nabla H(x_0)$, we have

$$\langle \frac{\partial x}{\partial x_0}^{*}(t,x_0)\,\nabla H(x_0),\, \nabla H\big(x(t,x_0)\big)\,\rangle = \|\nabla H(x_0)\|^2. \tag{4.5}$$

Two first integrals H_1, H_2 for (4.1) are said to be *independent* if $\nabla H_1(x)$ and $\nabla H_2(x)$ are linearly independent vectors in $\mathbf{R}^n$ for all $x \in D$. Suppose that (4.1) has r independent first integrals $H_1, \cdots, H_r$. A *level surface* will then be determined by r equations $H_i(x) = c_i$, $i = 1, \cdots, r$. We will denote it by $\Gamma(c_1, \cdots, c_r)$ or Γ when $c_1, \cdots, c_r$ are not essential. Since $\nabla H_i(x) \neq 0$ for each i is implied, we know that the level surface in this case is a $(n-r)$-dimensional smooth surface. In section §2, the first integrals are given by Bx, Γ is an affine surface, its normal vectors are the transpose of row vectors of the matrix B.

Let $Y(t)$ be the fundamental matrix of (4.3) such that $Y(0) = I$. For k elements $u_1, \cdots, u_k$ in $\mathbf{R}^n$, set

$$\Omega(t) =: Y(t)u_1 \wedge \cdots \wedge Y(t)u_k.$$

Using Proposition 6.4.1 above and Lemma 6.1.2 in §6.1, we can prove the following result. The proof is parallel to that of Proposition 6.1.5, the corresponding result in the affine case; it is sufficient only to replace w_i by $\nabla H_i(x_0)$ for each $1 \leq i \leq r$.

Proposition 6.4.2. *For any k elements $u_1, \cdots, u_k$ in $\mathbf{R}^n$, the corresponding function Ω satisfies*

$$\|\Omega(t)\| \leq \|Y(t)\nabla H_1(x_0) \wedge \cdots \wedge Y(t)\nabla H_r(x_0) \wedge \Omega(t)\| \tag{4.6}$$

for all $t \in \mathbf{R}$.

We would like to note that $z(t) = \Omega(t)$ and

$$w(t) = Y(t)\nabla H_1(x_0) \wedge \cdots \wedge Y(t)\nabla H_r(x_0) \wedge \Omega(t)$$

are solutions to the k-th and $(k+r)$-th compound equations of (4.1)

$$z'(t) = \frac{\partial f}{\partial x}^{[k]}\big(x(t,x_0)\big)\, z(t) \tag{4.7}$$

$$w'(t) = \frac{\partial f}{\partial x}^{[k+r]}\big(x(t,x_0)\big)\, w(t), \tag{4.8}$$

respectively.

Suppose $H_1, \cdots, H_r$ are independent. Let Γ be a level surface. For each $x_0 \in \Gamma$, we denote by $T_{x_0}\Gamma$ the tangent space of Γ at x_0. Then $T_{x_0}\Gamma$ is a r-dimensional subspace of $\mathbf{R}^n$, which is perpendicular to the subspace spanned by $\{\nabla H_1(x_0), \cdots, \nabla H_r(x_0)\}$, for each $x_0 \in \Gamma$. Note in the affine case, Γ is an affine surface, thus its tangent space is identical at every point and is given by $V_0 = \ker B$.

The following result follows from Proposition 6.4.2 and the definition of asymptotic stability.

Theorem 6.4.3. *Assume that* (4.1) *has* r *independent first integrals* $H_1, \cdots, H_r$ *and* Γ *is a level surface. For each* $x_0 \in \Gamma$, *and any* $u_1, \cdots, u_k \in T_{x_0}\Gamma$,

$$\lim_{t \to \infty} Y(t)u_1 \wedge \cdots \wedge Y(t)u_k = 0$$

if the linear system (4.8) *is asymptotically stable.*

Using Lozinskiĭ measure, we obtain the following concrete conditions from Theorem 6.4.3.

Corollary 6.4.4. *Under the assumptions of Theorem* 6.4.3, $Y(t)u_1 \wedge \cdots \wedge Y(t)u_k \to 0$ *as* $t \to \infty$, *if*

$$\lim_{t \to \infty} \int_{t_0}^{t} \mu \left(\frac{\partial f}{\partial x}^{[k+r]} (x(s, x_0)) \right) ds = -\infty. \tag{4.9}$$

Moreover, if $\mu\left(\frac{\partial f}{\partial x}^{[k+r]}(x(t, x_0))\right) \le -\delta < 0$, *then the convergence is exponential with an exponential rate* δ.

Our main object in this section is to derive conditions which preclude periodic solutions to an autonomous system (4.1) having independent first integrals $H_1, \cdots, H_r$. To this end, we have the following result.

Theorem 6.4.5. *Assume that* Γ *is a level surface determined by* r *independent first integrals for* (4.1). *Suppose*

(*a*) Γ *is simply connected,*

(*b*) Γ *is either bounded or contains a bounded absorbing set,*

(c) $\mu\left(\frac{\partial f}{\partial x}^{[r+2]}(x)\right) < 0$ *for all* $x \in \Gamma$.

Then no simple closed rectifiable curve in Γ *can be invariant with respect to* (4.1).

The theorem can be proved exactly the same as Theorem 6.3.5 is proved in the affine case. Suppose ψ is a simple closed rectifiable curve in Γ. Then for each

$\varphi \in \Sigma(\psi, \Gamma)$, $\varphi_t = x(t, \varphi) \in \Sigma(\psi, \Gamma)$ for all $t \in \mathbf{R}$. Now (a) (b) imply that

$$\frac{\partial \varphi_t}{\partial p_1} \wedge \frac{\partial \varphi_t}{\partial p_2} \longrightarrow 0, \qquad \text{as} \quad t \longrightarrow \infty$$

uniformly for $p = (p_1, p_2) \in B$, the unit ball in $\mathbf{R}^2$, by Corollary 6.4.4. Therefore $\mathcal{A}\varphi_t \to 0$ as $t \to \infty$. Again this is impossible by Proposition 6.2.4.

Remarks.

(i) Generalizations of Theorem 6.4.5 in terms of Dulac-type conditions can be formed as discussed in the remark following Theorem 6.2.5 in §6.2.

(ii) When $r = n - 2$, $r + 2 = n$ and condition (b) becomes

$$\operatorname{div} f < 0. \tag{4.10}$$

Suppose now $r = n - 2$, then $n - r = 2$. Therefore a level surface Γ is a 2-dimensional smooth surface in $\mathbf{R}^n$. The portion S of Γ enclosed by the simple closed invariant curve ψ is bounded and invariant. Moreover, a 2-surface $\varphi \in \Sigma(\psi, \Gamma)$ can be chosen so that its trace $\varphi(\overline{B})$ is S, and $\mathcal{A}\varphi_t$ is the 2-dimensional surface area of S. Thus $\mathcal{A}\varphi_t =$ constant > 0. Now the condition (4.10) holding almost everywhere on Γ will, then, imply $\mathcal{A}\varphi_t \to 0$ as $t \to \infty$. Thus the theorem can be proved under a weaker condition and without the assuming Γ is bounded. This discussion gives rise to the following corollary.

Corollary 6.4.6. *Assume that* (4.1) *has* $n - 2$ *independent first integrals, and that a level surface* Γ *is simply connected. Suppose* (4.10) *holds almost everywhere on* Γ. *Then no simple closed rectifiable curve in* Γ *can be invariant with respect to* (4.1).

Remark. Demidovich [3] proved a similar result assuming $n = 3$, $r = 1$. Using a different analysis involving the cross product in $\mathbf{R}^3$, he shows that no periodic solutions can exist on Γ if (4.11) holds almost everywhere on Γ. He also assume C^2 smoothness on the level surface.

Results concerning other nonlinear problems such as orbital stability for periodic orbits and stability at large for equilibria on level surfaces can also be formulated and proved similarly as in the previous sections. We will not discuss them here.

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CHAPTER VII

APPLICATIONS IN MATHEMATICAL BIOLOGY

In this chapter, the general theory developed in previous chapters will be applied to resolve the question of global stability for some epidemiological models, which has been a long-standing open problem in Mathematical Biology. Ordinary differential equations have long been used to model the epidemics of infectious diseases in certain populations (of humans, flocks, bacteria, etc.). The epidemiological models have evolved from the earlier SIR, SIRS to the later ones such as SEI and SEIS. The latest SEIR and SEIRS models receive considerable attention. The SEIRS model is described by the following system of equations

$$\begin{aligned} S' &= -\lambda I^p S^q + \nu - \nu S + \delta R \\ E' &= \lambda I^p S^q - (\epsilon + \nu) E \\ I' &= \epsilon E - (\gamma + \nu) I \\ R' &= \gamma I - (\nu + \delta) R \end{aligned}$$

where $p, q, \gamma, \nu, \lambda, \delta,$ and ϵ are nonnegative parameters and $S, E, I,$ and R denote the fractions of the population that are susceptible, exposed, infectious, and recovered, respectively. Some notable features of the model: the birth rate and the death rate are assumed to be equal (denoted by ν) and in consequence the total population is at an equilibrium; the incidence rate (the rate of new infections) is described by the nonlinear term $\lambda I^p S^q$ which includes the traditional bilinear case $(p = q = 1)$; a latent period is introduced on the basis of the SIS and SIR models. Individuals are susceptible, then exposed (in the latent period), then infectious, then recovered with the possibility of becoming susceptible again. If the immunity is assumed to be permanent, which amounts to assuming $\delta = 0$ in the equations, the SEIRS becomes the SEIR model. In this case, individuals, once infected and

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recovered, will not become susceptible again. This is quite typical for some diseases such as measles and varicella.

Good surveys on the development of epidemiological models can be found in a recent book by J. D. Murray [11] and a survey paper by H. W. Hethcote, H. W. Stech, and P. van den Driessche [4]. A detailed analysis on the SEIRS models was given in 1986 by W.-M. Liu, H. W. Hethcote, and S. A. Levin [9], where local stability of equilibria and possible Hopf bifurcations are carefully investigated for a wide range of parameters. We will outline some of their results in the following. The proofs and more detailed study can be found in [9].

The variables S, E, I, R are assumed to be nonnegative, which means the feasible region for the SEIRS is the positive cone $\mathbf{R}_+^4$ of $\mathbf{R}^4$. Adding all the equations, we have

$$(S + E + I + R)' = -\nu(S + E + I + R - 1),$$

which has the following implications: the 3-dimensional simplex

$$\Gamma = \{(S, E, I, R) \in \mathbf{R}_+^4 \ : \ S + E + I + R = 1\}$$

is positively invariant; the system is dissipative and the global attractor is contained in $\overline{\Gamma}$. Moreover, it suffices to study the dynamics on the simplex $\overline{\Gamma}$.

When $0 < p \leq 1$, there are two possible equilibria: the disease-free equilibrium $P_0 = (1, 0, 0, 0)$ and the endemic equilibrium P^*. When $0 < p < 1$, P_0 is unstable and all solutions starting near P_0 except those from the face of Γ on the coordinate plane $E = I = 0$ move away from P_0; P^* is in the interior of Γ and is locally asymptotically stable. When $p = 1$, the contact number $\sigma = \lambda\epsilon/(\epsilon + \nu)(\gamma + \nu)$ satisfies a threshold condition: if $\sigma \leq 1$, P_0 is the only equilibrium in $\overline{\Gamma}$ and is globally asymptotically stable; if $\sigma > 1$, P_0 becomes an unstable saddle with one of the two unstable eigenvectors pointing to the inside of Γ while P^* emerges as a locally asymptotically stable equilibrium in the interior of Γ. It was conjectured in [9] that P^* is globally asymptotically stable whenever it belongs to the interior of Γ; namely, when $0 < p < 1$, or $p = 1$ and $\sigma > 1$.

This conjecture on the global stability has caught the attention of many mathematicians and mathematical biologists. However, it remained an open problem over the years. A claim by Rinaldi [12] to have resolved the problem when $p = q = 1$ proved to be erroneous. In this chapter, using our new geometric approach developed in Chapter V and our method of dealing with systems having an invariant linear subspace discussed in Chapter VI, we will give this conjecture an affirmative answer.

The strategies we use to deal with the SEIR and the SEIRS are quite different. Both of them are new for this type of problem and both have the potential to become useful general approaches to the problem of global stability.

The SEIR model, the simpler of the two, is analyzed in the section §7.1. It has been observed by H. L. Smith, P. Waltman and others that the SEIR can be reduced to a 3-dimensional order-preserving system, and thus possesses the Poincaré-Bendixson property (see [5] or [13]). The key to proving the global stability is then to rule out periodic orbits. This is done here by showing that any periodic orbit of the SEIR, if there is one, is locally attracting. A simple application of the Poincaré-Bendixson property can show that, if any periodic orbits exist, this will lead to a contradiction, because of the local attractivity of P^*. This is an unusual tactic. It is introduced in the work of G. J. Butler, S. B. Hsu and P. Waltman [2] on a two dimensional autonomous system reduced from a chemostat model. Their way to show the local attractivity for periodic orbits is to apply the orbital stability criterion of Poincaré for planar systems (see Chapter IV). It did not receive much attention after the work of Butler, Hsu and Waltman partly because of the lack of higher dimensional generalizations of Poincaré's criterion at the time. As is mentioned in Chapter IV, this higher dimensional generalization has been developed by J. S. Muldowney [10]. We believe that, with these orbital stability criteria gradually becoming well known, this tactic will prove to be a very useful tool in the analysis of stability problems for systems possessing the Poincaré-Bendixson property.

It is not obvious that the SEIRS model can be reduced to a system having the Poincaré-Bendixson property. Our strategy for the SEIRS then is to use our new geometric approach based on our autonomous convergence theorems developed in Chapter V. The theory on the autonomous systems having an invariant linear subspace established in Chapter VI also plays an essential role in the analysis. These will be discussed in the section §7.2.

Incidence rate—the rate of infection—plays a very important role in these epidemiological models. We would like to remark that our method can deal with SEIR and SEIRS models with very general forms of incidence rate without additional difficulty. This suggests that they have advantages over the usual approach of constructing Lyapunov functions, in which one Lyapunov function usually only works for a specific type of nonlinearity.

§7.1. Global Stability of the SEIR Models in Epidemiology

The SEIR model in Epidemiology for the spread of an infectious disease is described by the following system of differential equations:

$$\begin{aligned} S' &= -\lambda I^p S^q + \nu - \nu S \qquad (1.1) \\ E' &= \lambda I^p S^q - (\epsilon + \nu) E \\ I' &= \epsilon E - (\gamma + \nu) I \\ R' &= \gamma I - \nu R \end{aligned}$$

Throughout this section, we shall assume that $0 < p < 1$.

We have seen in the preliminary analysis given earlier that the 3-dimensional simplex in $\mathbf{R}^4_+$

$$\Gamma = \{(S, E, I, R) \in \mathbf{R}^4_+ \; : \; S + E + I + R = 1\}$$

is positively invariant with respect to (1.1), and it suffices to study the dynamics of

(1.1) on the simplex Γ.

On Γ,

$$R(t) = 1 - S(t) - E(t) - I(t).$$

Thus (1.1) reduces to the following 3-dimensional system:

$$S' = -\lambda I^p S^q + \nu - \nu S$$

$$E' = \lambda I^p S^q - (\epsilon + \nu) E$$

$$I' = \epsilon E - (\gamma + \nu) I \tag{1.2}$$

The dynamical behaviour of (1.1) on Γ is equivalent to that of (1.2). Therefore in the rest of the section we will study the system (1.2) in the region

$$T = \{(S, E, I) \; : \; 0 \leq S, E, I \leq 1, \; S + E + I \leq 1\}, \tag{1.3}$$

and formulate our results accordingly.

The main aim of the section is to prove the following result:

Theorem 7.1.1. *If* $0 < p < 1$ *or* $p = 1$ *and* $\sigma > 1$, *the endemic equilibrium* P^* *is globally asymptotically stable in the interior of* T.

Remark. Once Theorem 7.1.1 is proved, the global dynamical behaviour of (1.1) is completely determined when $0 < p \leq 1$.

The proof of Theorem 7.1.1 will be given at the end of the section. At first, we prove some basic properties of the system (1.2) that will be used later.

Proposition 7.1.2. *The disease-free equilibrium* P_0 *is the only omega limit point of* (1.2) *on the boundary of* T.

Proof. It is easy to see that the vector field of (1.2) is transversal to the boundary of T on all its faces except the S-axis which is invariant with respect to (1.2). On the S-axis, the equation S satisfies is $S' = \nu - \nu S$, which implies that $S(t) \to 1$,

as $t \to \infty$. Therefore P_0 is the only omega limit point on the boundary of T.
$\square$

Proposition 7.1.3. *Suppose $0 < p < 1$ or $p = 1$ and $\sigma > 1$. Then P_0 can not be the omega limit point of any orbit starting in the interior of T.*

Proof. Consider the function

$$L = E + \frac{\epsilon + \nu}{\epsilon} I.$$

Its derivative along the solutions of (1.2) is

$$L' = \lambda I^p \left(S^q - \frac{1}{\sigma} I^{1-p}\right).$$

Suppose now $p < 1$ or $p = 1$ and $\sigma > 1$. In the feasible region close enough to P_0, we always have $L' > 0$ as long as $I > 0$. Therefore P_0 can only be the omega limit point of orbits on the invariant S-axis. Thus the lemma is proved. $\square$

Remark. From Proposition 7.1.2 and Proposition 7.1.3 we know that, when $0 < p < 1$ or $p = 1$ and $\sigma > 1$, the system (1.2) is persistent in the sense described in [1].

Our next result establishes the local attractivity for periodic orbits of (1.2), when they exist. The proof uses the orbital stability criteria of J. S. Muldowney for periodic orbits presented in Chapter IV.

Theorem 7.1.4. *The trajectory of any nonconstant periodic solution to (1.2), if it exists, is asymptotically orbitally stable with asymptotic phase.*

Proof. The Jacobian matrix $J(S, E, I)$ of (1.2) is given by

$$J(S,E,I) = \begin{bmatrix} -\lambda q I^p S^{q-1} - \nu & 0 & -\lambda p I^{p-1} S^q \\ \lambda q I^p S^{q-1} & -\epsilon - \nu & \lambda p I^{p-1} S^q \\ 0 & \epsilon & -\gamma - \nu \end{bmatrix}. \tag{1.4}$$

The second compound equation of (1.4) is given by the following 3×3 system:

$$\begin{aligned} X' &= -(\lambda q I^p S^{q-1} + \epsilon + 2\nu) X + \lambda p I^{p-1} S^q (Y + Z) \\ Y' &= \epsilon X - (\lambda q I^p S^{q-1} + \gamma + 2\nu) Z \\ Z' &= \lambda q I^p S^{q-1} Y - (\epsilon + \gamma + 2\nu) Z \end{aligned} \tag{1.5}$$

To show the asymptotic stability of the system (1.5) we choose the following function:

$$V(X, Y, Z;\ S, E, I) = |P(S, E, I)(X, Y, Z)^*| \tag{1.6}$$

where the matrix $P = \operatorname{diag}(1, E/I, S/qI)$, $(X, Y, Z)^*$ is the transpose of the row vector (X, Y, Z), and $|\cdot|$ is the norm in $\mathbf{R}^3$ defined by

$$|(X, Y, Z)| = \max\{|X|,\ |Y|,\ |Z|\}. \tag{1.7}$$

Suppose that the solution $(S(t), E(t), I(t))$ of (1.2) is periodic of least period $\omega > 0$. Then from Proposition 7.1.2, its orbit γ remains at a positive distance from the boundary of T. The matrix P and its inverse are thus well defined and smooth along γ. There is a constant $c > 0$ such that

$$V(X, Y, Z; S, E, I) \geq c|(X, Y, Z)| \tag{1.8}$$

for all $(X, Y, Z) \in \mathbf{R}^3$ and $(S, E, I) \in \gamma$. Let $(X(t), Y(t), Z(t))$ be a solution of (1.5) and

$$V(t) = V\big(X(t), Y(t), Z(t);\ S(t), E(t), I(t)\big).$$

The right-hand derivative of $V(t)$ exists and its calculation is described in the Appendix A. Now we have by differentiation

$$D_t^+ V(t) \leq \sup\{g_1, g_2, g_3\}\, V(t), \tag{1.9}$$

where

$$g_1 = -\lambda q I^p S^{q-1} - \epsilon + p\frac{\lambda I^p S^q}{E} + \lambda p q I^p S^{q-1} - 2\nu,$$

$$g_2 = \frac{\mathrm{d}}{\mathrm{dt}}(\log \frac{E}{I}) - \lambda q I^p S^{q-1} - \gamma + \frac{\epsilon E}{I} - 2\nu,$$

$$g_3 = \frac{\mathrm{d}}{\mathrm{dt}}(\log \frac{S}{I}) - \epsilon - \gamma + \frac{\lambda I^p S^q}{E} - 2\nu.$$

In fact $\sup\{g_1, g_2, g_3\}$ is the *Lozinskiĭ measure* with respect to the vector norm $|\cdot|$ in $\mathbf{R}^3$ (see Appendix A).

From the second and last equations in (1.2) we have the following:

$$\frac{\lambda I^p S^q}{E} = \frac{E'}{E} + \epsilon + \nu, \tag{1.10}$$

$$\frac{\epsilon E}{I} = \frac{I'}{I} + \gamma + \nu. \tag{1.11}$$

From (1.10) we find

$$\begin{aligned}\int_0^\omega g_3 \; dt &= \int_0^\omega \left\{\frac{\mathrm{d}}{\mathrm{dt}}(\log \frac{S}{I}) - \gamma - \nu \frac{E'}{E}\right\} dt \\ &= -(\gamma+\nu)\omega + \left\{\log \frac{S}{I} - \log E\right\}\Big|_0^\omega \\ &= -(\gamma+\nu)\omega < -\nu\omega,\end{aligned}$$

(1.11) implies

$$\begin{aligned}\int_0^\omega g_2 \; dt &= \int_0^\omega \left\{\frac{\mathrm{d}}{\mathrm{dt}}(\log \frac{E}{I}) - \lambda q I^p S^{q-1} - \nu + \frac{I'}{I}\right\} dt \\ &\le -\nu\omega + \left\{\log \frac{E}{I} + \log I\right\}\Big|_0^\omega \le -\nu\omega,\end{aligned}$$

and (1.10) leads to

$$\begin{aligned}\int_0^\omega g_1 \; dt &= \int_0^\omega \left\{(p-1)\lambda q I^p S^{q-1} + (p-1)\epsilon + (p-2)\nu + \frac{pE'}{E}\right\} dt \\ &\le \{(p-1)\epsilon + (p-2)\nu\}\,\omega + p\log E|_0^\omega \le -\nu\omega.\end{aligned}$$

Therefore

$$\int_0^{\omega} \sup\{g_1, g_2, g_3\}\, dt \leq -\nu\omega < 0.$$

Since each g_i is ω-periodic in t, this and the inequality (1.9) imply that $V(t) \to 0$ as $t \to \infty$. Thus $(X(t), Y(t), Z(t)) \to 0$ as $t \to \infty$ from (1.8). As a result, the linear system (1.5) is asymptotically stable and the orbit γ of the periodic solution $(S(t), E(t), I(t))$ is asymptotically orbitally stable with asymptotic phase by Theorem 4.4.2 of Chapter IV. □

Before presenting the proof of Theorem 7.1.1, we will study further dynamical behaviour of the SEIR model (1.2).

An autonomous system $x' = f(x)$ in $\mathbf{R}^n$ is said to be *monotone* if, for some diagonal matrix $H = \operatorname{diag}(\epsilon_1, \cdots, \epsilon_n)$ where each ϵ_i is either 1 or -1, $H\partial f/\partial x H$ has nonnegative off diagonal elements. It is proved in [13] that the flow of such a system preserves the partial order in $\mathbf{R}^n$ defined by the orthant

$$K = \{(x_1, \cdots, x_n) \in \mathbf{R}^n \;:\; \epsilon_i x_i \geq 0,\ i = 1, \cdots, n\}.$$

It is easy to check by considering the Jacobian matrix of the system (1.2) that this system is monotone in $\mathbf{R}^3$ and the corresponding orthant is

$$\{(S, E, I) \in \mathbf{R}^3 \;:\; S \leq 0,\ E \geq 0,\ I \leq 0\}.$$

It is known that such systems have the Poincaré-Bendixson property [5] [13].

Theorem 7.1.5. *For an irreducible monotone system in* $\mathbf{R}^3$, *any compact omega limit set containing no equilibria is a closed orbit (see* [13], *Theorem* 2.4 *).*

Using Theorem 7.1.5 we can show that the system (1.2) possesses the following strong Poincaré-Bendixson property.

Theorem 7.1.6. *Any compact omega limit set of* (1.2) *in the interior of* T *is either a closed orbit or the endemic equilibrium* P^*.

Proof. Suppose that Ω is an omega limit set of (1.2) in the interior of T. If Ω does not contain P^*, then it contains no equilibria since P^* is the only interior equilibrium. Theorem 7.1.5 will then imply that Ω is a closed orbit. Suppose Ω contains P^*. Since P^* is asymptotically stable whenever it exists in the interior of T, any orbit that gets arbitrarily close to P^* must converge to P^*. Thus $\Omega = \{P^*\}$. □

Now we are ready to present the proof for Theorem 7.1.1.

Proof of Theorem 7.1.1. The basin of attraction U of the endemic equilibrium P^* is a relatively open subset of T since P^* is locally asymptotically stable. If U is not all of T, then its boundary ∂U has nonempty intersection with the interior of T, we denote this intersection by Σ. Now Σ is obviously invariant and thus $\bar{\Sigma}$ contains a nonempty compact omega limit set γ which is in the interior of T by Proposition 7.1.2 and Proposition 7.1.3. Moreover, γ obviously does not contain P^* and thus contains no equilibria. We can then conclude from Theorem 7.1.6 and Theorem 7.1.4 that γ is a nontrivial periodic orbit which is asymptotically orbitally stable with asymptotic phase. But this contradicts the fact that Σ, hence γ, is contained in the alpha limit set of P^*. The contradiction shows that U is the interior of T and thus P^* is globally asymptotically stable. □

§7.2. Global Stability of the SEIRS Models

In this section, we consider the SEIRS models with $\delta > 0$. The equations are

$$S' = -\lambda I^p S^q + \nu - \nu S + \delta R \tag{2.1}$$

$$E' = \lambda I^p S^q - (\epsilon + \nu) E$$

$$I' = \epsilon E - (\gamma + \nu) I$$

$$R' = \gamma I - (\nu R + \delta R)$$

Because of the δR term in the first equation, if we again try to eliminate R using $R = 1 - S - E - I$, the resulting 3-dimensional system does not necessarily preserve an order in $\mathbf{R}^3$. Therefore the Poincaré-Bendixson property, which is crucial to the analysis of the SEIR in §7.1, will no long hold in this situation.

The idea then is to use the geometric approach based on the autonomous convergence theorems developed in the Chapter V, and the fact that the SEIRS has an invariant 1-dimensional linear subspace.

The preliminary analysis of (2.1) has been given in the summary at the beginning of the chapter. The dynamics of (2.1) will be analyzed in the following invariant 3-dimensional simplex in $\mathbf{R}^4_+$

$$\Gamma = \{(S, E, I, R) \in \mathbf{R}^4_+ \ : \ S + E + I + R = 1\}. \tag{2.2}$$

The basic assumption on the parameters is

$$\textit{either} \quad 0 < p < 1 \quad \textit{or} \quad p = 1 \quad \textit{and} \quad \sigma > 1,$$

where $\sigma = \lambda\epsilon/(\epsilon + \nu)(\gamma + \nu)$ is the contact number when $p = 1$.

The aim of this section is to prove the following result.

Theorem 7.2.1. *If $0 < p < 1$ or $p = 1$ and $\sigma > 1$, the endemic equilibrium P^* of (2.1) is globally asymptotically stable in the interior of Γ.*

Remark. As in the case of the SEIR, once Theorem 7.2.1 is proved, the global dynamical behaviour of (2.1) is completely determined when $0 < p \leq 1$.

The following lemma is important.

Lemma 7.2.2. *Under the assumptions of Theorem 7.2.1, there is a positive constant $0 < c < 1$, such that for any solution $(S(t), E(t), I(t), R(t))$ with $(S(0), E(0),$*

$I(0), R(0))$ *in the interior of* Γ,

$$\liminf_{t\to\infty} S(t), E(t), I(t), R(t) \geq c > 0,$$

$$\liminf_{t\to\infty} S(t) + E(t) + I(t) + R(t) \leq 1 - c > 0.$$

Remark. This lemma says that, under the assumptions of Theorem 7.2.1, (2.1) is uniformly persistent in the sense of [1].

Corollary 7.2.3. *Under the assumptions of Theorem* 7.2.1, (2.1) *has in the interior of* Γ *a compact absorbing set* D_0.

Proof of Lemma 7.2.2. Using $S + E + I + R = 1$, (2.1) can be reduced to the following 3-dimensional system

$$\begin{aligned} E' &= \lambda I^p (1 - E - I - R)^q - (\epsilon + \nu) E \\ I' &= \epsilon E - (\gamma + \nu) I \\ R' &= \gamma I - (\nu R + \delta R) \end{aligned} \tag{2.3}$$

with (E, I, R) in the feasible region

$$\Delta = \{(E, I, R) \in \mathbf{R}^3_+ \ : \ E + I + R \leq 1\}.$$

Now $(0,0,0)$ is the trivial equilibrium which is unstable with a 2-dimensional stable manifold. Moreover, all trjectories starting in Δ near $(0,0,0)$ except those starting on the R-axis leaves $(0,0,0)$ (see [9]). It is also easy to check from these equations that the vector field points strictly inwards Δ at every point on the boundary of Δ except on the R-axis. By a result of J. Hofbauer and J. W.-H. So (Theorem 4.1 [6]), we know that (2.2), hence (2.1), is uniformly persistent. Thus the lemma is proved.

Proof of Theorem 7.2.1. The Jacobian matrix $J = J(S, E, I, R)$ of (2.1) is given by

$$J = -\nu I_{4\times 4} + \begin{bmatrix} -\lambda q I^p S^{q-1} & 0 & -\lambda p I^{p-1} S^q & \delta \\ \lambda q I^p S^{q-1} & -\epsilon & \lambda p I^{p-1} S^q & 0 \\ 0 & \epsilon & -\gamma & 0 \\ 0 & 0 & \gamma & -\delta \end{bmatrix}. \tag{2.4}$$

Let Φ denote the second part of the decomposition in (2.3) and let $B = (1,1,1,1)$. Then $B\Phi = 0$ in $\mathbf{R}^4_+$. Therefore (2.1) satisfies the assumptions $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$ in §6.2, Chapter VI, with $D = \mathbf{R}^4_+$, $B = (1,1,1,1)$, $r = 1$, $\nu = \nu$, and the invariant affine manifold Γ as given in (2.2). Theorem 6.2.11 in §6.2, Chapter VI suggests that we consider the third additive compound matrix of J, which is given by (see Appendix B)

$$J^{[3]} = -3\nu I_{4\times 4} + \Phi^{[3]}$$

and

$$\Phi^{[3]} = \tag{2.5}$$

$$\begin{bmatrix} -\lambda q I^p S^{q-1} - \epsilon - \gamma & 0 & 0 & \delta \\ \gamma & -\lambda q I^p S^{q-1} - \epsilon - \delta & \lambda p I^{p-1} S^q & \lambda p I^{p-1} S^q \\ 0 & \epsilon & -\lambda q I^p S^{q-1} - \gamma - \delta & 0 \\ 0 & 0 & \lambda q I^p S^{q-1} & -\epsilon - \gamma - \delta \end{bmatrix}.$$

Consider a 4×4 diagonal matrix

$$A = \operatorname{diag}\left(\frac{\alpha R}{I}, 1, \frac{E}{I}, \frac{S}{qI}\right) \tag{2.6}$$

where $\alpha > 1$ is chosen so that

$$(\alpha - 1)\lambda q < \epsilon + \gamma. \tag{2.7}$$

The matrix A is C^1 and is nonsingular in the interior of Γ. We also want to set

$$\beta = \max\{2\nu, (1-p)\epsilon + (-\frac{1}{\alpha} + 1)\delta, \gamma + \delta + \nu\}. \tag{2.8}$$

Let f denote the vector field of (2.1). We want to compute the matrix

$$B = A_f A^{-1} + A \frac{\partial f}{\partial x}^{[3]} A^{-1} \tag{2.9}$$

considered in (2.20) of the Remark (ii) following the proof of Theorem 6.2.5 in §6.2, Chapter VI, for $n = 4$, $r = 1$, $\frac{\partial f}{\partial x} = J$. Note that, in this case, $r + 2 = 3$, $N = \binom{n}{r+2} = \binom{4}{3} = 4$.

Direct calculation gives us

$$A_f = \operatorname{diag}\big((\frac{\alpha R}{I})_f,\ 0,\ (\frac{E}{I})_f,\ (\frac{S}{qI})_f \big) \tag{2.10}$$

$$A \frac{\partial f}{\partial x}^{[3]} A^{-1} = -3\nu I_{4\times 4} + A\,\Phi^{[3]} A^{-1}$$

and

$$A\Phi^{[3]} A^{-1} = \tag{2.11}$$

$$\begin{bmatrix} -\lambda q I^p S^{q-1} - \epsilon - \gamma & 0 & 0 & \alpha q \frac{\delta R}{S} \\ \frac{1}{\alpha}\frac{\gamma I}{R} & -\lambda q I^p S^{q-1} - \epsilon - \delta & p\frac{\lambda I^p S^q}{E} & pq\lambda I^p S^{q-1} \\ 0 & \frac{\epsilon E}{I} & -\lambda q I^p S^{q-1} - \gamma - \delta & 0 \\ 0 & 0 & \frac{\lambda q I^p S^q}{E} & -\epsilon - \gamma - \delta \end{bmatrix}.$$

We choose the norm in $\mathbf{R}^4 \cong \mathbf{R}^{\binom{4}{3}}$ as

$$|(u, v, w, z)| = \max\{|u|, |v|, |w|, |z|\}. \tag{2.12}$$

Then the corresponding Lozinskiĭ measure $\mu(B)$ can be calculated, according to Proposition A.2.3 of Appendix A, as follows,

$$\mu(B) = \max\{g_1, g_2, g_3, g_4\} \tag{2.13}$$

where g_1, g_2, g_3, and g_4 are given by

$$g_1 = -\lambda q I^p S^{q-1} - \epsilon - \gamma + \alpha q \frac{\delta R}{S} + (\frac{\alpha R}{I})_f - 3\nu$$

$$g_2 = -\lambda q I^p S^{q-1} - \epsilon - \delta + \frac{1}{\alpha}\frac{\gamma I}{R} + p\frac{\lambda I^p S^q}{E} + pq\lambda I^p S^{q-1} - 3\nu$$

$$g_3 = -\lambda q I^p S^{q-1} - \gamma - \delta + \frac{\epsilon E}{I} + (\frac{E}{I})_f - 3\nu$$

$$g_4 = -\epsilon - \gamma - \delta + \frac{\lambda q I^p S^q}{E} + (\frac{S}{qI})_f - 3\nu.$$

Along each solution of (2.1), we have, from the equations of (2.1),

$$\frac{\delta R}{S} = \frac{S'}{S} + \lambda I^p S^{q-1} - \frac{\nu(1-S)}{S}, \tag{2.14}$$

$$\frac{\lambda I^p S^q}{E} = \frac{E'}{E} + \epsilon + \nu, \tag{2.15}$$

$$\frac{\epsilon E}{I} = \frac{I'}{I} + \gamma + \nu, \tag{2.16}$$

$$\frac{\gamma I}{R} = \frac{R'}{R} + \delta + \nu. \tag{2.17}$$

Now (2.15) implies

$$\begin{aligned} \frac{1}{t}\int_0^t g_4 &= \frac{1}{t}\int_0^t \left\{-\gamma - \delta + \frac{E'}{E} - 2\nu + \frac{\mathrm{d}}{\mathrm{dt}}(\frac{S}{qI})\right\} \\ &= -\gamma - \delta - 2\nu + \frac{1}{t}\left\{\ln\frac{E(t)}{E(0)} + \frac{S(t)}{qI(t)} - \frac{S(0)}{qI(0)}\right\} \end{aligned}$$

Now Lemma 7.2.2 implies that

$$\lim_{t\to\infty} \frac{1}{t}\left\{\ln\frac{E(t)}{E(0)} + \frac{S(t)}{qI(t)} - \frac{S(0)}{qI(0)}\right\} = 0.$$

Therefore, for all sufficiently large $t > 0$,

$$\frac{1}{t}\int_0^t g_4 \le -\gamma - \delta - 2\nu \le -\nu - \beta. \tag{2.18}$$

Similarly, using (2.16), we have

$$\begin{aligned}\frac{1}{t}\int_0^t g_3 &= \frac{1}{t}\int_0^t \left\{-\lambda q I^p S^{q-1} - \delta - 2\nu + \frac{I'}{I} + \frac{\mathrm{d}}{\mathrm{dt}}(\frac{E}{I})\right\} \\ &< -\delta - 2\nu + + \frac{1}{t}\left\{\ln\frac{I(t)}{I(0)} + \frac{E(t)}{I(t)} - \frac{E(0)}{I(0)}\right\} \\ &< -\nu - \beta \qquad (2.19)\end{aligned}$$

for all sufficiently large $t > 0$. From (2.14) we arrive at

$$\begin{aligned}\frac{1}{t}\int_0^t g_1 &= \frac{1}{t}\int_0^t \left\{(\alpha-1)\lambda q I^p S^{q-1} - \epsilon - \gamma - \frac{\alpha q \nu(1-S)}{S}\right. \\ &\quad \left. + \alpha q \frac{S'}{S} + \frac{\mathrm{d}}{\mathrm{dt}}(\frac{\alpha R}{I}) - 3\nu\right\} \\ &< \big((\alpha-1)\lambda q - \epsilon - \gamma - 3\nu\big) + \frac{1}{t}\left\{\alpha q \ln\frac{S(t)}{S(0)} + \frac{\alpha R(t)}{I(t)} - \frac{\alpha R(0)}{I(0)}\right\} \\ &< -\nu - \beta \qquad (2.20)\end{aligned}$$

for all sufficiently large $t > 0$. Finally, (2.17) implies

$$\begin{aligned}\frac{1}{t}\int_0^t g_2 &= \frac{1}{t}\int_0^t \left\{(p-1)\lambda q I^p S^{q-1} + (p-1)\epsilon + (\frac{1}{\alpha} - 1)\delta\right. \\ &\quad \left. + (\frac{1}{\alpha} + p)\nu + p\frac{E'}{E} + \frac{1}{\alpha}\frac{R'}{R} - 3\nu\right\} \\ &\leq (p-1)\epsilon + (\frac{1}{\alpha} - 1)\delta + (\frac{1}{\alpha} - 2)\nu + \frac{1}{t}\left\{\ln\frac{E(t)}{E(0)} + \frac{1}{\alpha}\ln\frac{R(t)}{R(0)}\right\} \\ &< -\nu - \beta \qquad (2.21)\end{aligned}$$

for all sufficiently large $t > 0$.

Now the quantities $\bar{q}_3(t, D_0)$ and $\bar{q}_3(D_0)$, defined in (2.18) and (2.19) of Chapter VI for $r = 1$, can be estimated using (2.18) – (2.21) as follows

$$\bar{q}_3(t, D_0) =: \sup_{\bar{D}_0} \frac{1}{t}\int_0^t \mu(B) \leq -\nu - \beta,$$

and

$$\bar{q}_3(D_0) =: \liminf_{t\to\infty} \bar{q}_3(t, D_0) < -\nu.$$

Since (2.1) satisfies $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$ and has an absorbing set D_0 in Γ, $\bar{q}_3(D_0) < -\nu$ implies P^* is globally asymptotically stable in the interior of Γ, using Theorem 6.2.11 in Chapter VI, and the remarks follow it. Therefore the theorem is proved. □

Remark. The same strategies and techniques used in §7.1 and §7.2 can also be applied to solve similar problems in other types of epidemiological models such as SEIRS with more general nonlinear forms of incidence rate, and SEIRS models with variable total populations. However, we will not discuss them here.

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APPENDIX A

INDUCED MATRIX NORMS AND THE LOZINSKIĬ MEASURE

Studies on the exponential growth of a general vector norm $|x(t)|$ of solutions $x(t)$ to a linear system of ordinary differential equations

$$x'(t) = A(t)\,x(t)$$

play an essential role in the investigation of many nonlinear problems addressed in this thesis. When $|\cdot|$ is the l_2 norm, it is well-known that the exponential growth rate of $|x(t)|$ can be determined by the largest eigenvalue of the matrix $(A^*(t) + A(t))/2$, the symmetric part of $A(t)$. It turns out that, for a general vector norm $|\cdot|$, the exponential rate of $|x(t)|$ depends on $\mu(A(t))$ — the *Lozinskiĭ measure* of $A(t)$ (sometimes $\mu(A(t))$ is also called the *Logarithmic norm* of $A(t)$). Eigenvalues are known to be difficult to compute, whereas the Lozinskiĭ measure can provide flexibility in calculation. This is the main reason for the extensive use of Lozinskiĭ measure in the thesis. In this appendix, we review the theory on the Lozinskiĭ measure and provide the technical details not included in the text.

In Section A.1, definitions and properties of induced matrix norms are reviewed; in Section A.2, the Lozinskiĭ measure is defined and its properties and calculations are discussed; in Section A.3, exponential growth of $|x(t)|$ is studied using the Lozinskiĭ measure. Its applications to the stability of linear systems of ordinary differential equations are presented in Section A.4.

§A.1. Induced Matrix Norms

Given a vector norm $|\cdot|$ in $\mathbf{R}^n$, there is an induced norm in $\mathbf{R}^{n\times n}$, the space of all $n \times n$ matrices defined by

$$|A|_i = \sup_{\substack{x\neq 0\\ x\in\mathbf{R}^n}} \frac{|Ax|}{|x|} \tag{1.1}$$

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where the subscript 'i' indicates the induced norm. In what follows we shall always use capital letters for matrices so that this subscript can be omitted. From the homogeneity of vector norms and the finite dimensionality of $\mathbf{R}^n$ we can easily see that

$$|A| = \sup_{|x|\leq 1} \frac{|Ax|}{|x|} = \sup_{|x|=1} |Ax|. \tag{1.2}$$

Remark. Not all vector norms in $\mathbf{R}^{n\times n}$ can be induced from some vector norm in $\mathbf{R}^n$ using (1.1). Let $A = (a_{ij})_{n\times n}$ be a $n \times n$ matrix, define

$$|A| = \max_{1\leq i,j\leq n} |a_{ij}|.$$

Then $|\cdot|$ is a vector norm in $\mathbf{R}^{n\times n}$. It can not, however, be an induced matrix norm since it does not satisfy the submultiplicative property given in the following Proposition A.1.1.

Proposition A.1.1. *An induced matrix norm satisfies the submultiplicative property:*

$$|AB| \leq |A|\,|B|. \tag{1.3}$$

Proof. For any $x \in \mathbf{R}^n$, we have

$$|Ax| \leq |A|\,|x|, \quad \text{and} \quad |Bx| \leq |B|\,|x|.$$

Thus

$$|ABx| \leq |A(Bx)| \leq |A|\,|Bx| \leq |A|\,|B|\,|x|,$$

and this implies $|AB| \leq |A|\,|B|$. □

Some common norms in $\mathbf{R}^n$ are l_∞, l_1, and l_2 norm given by:

$$\max\{\,|x_j| \;:\; 1 \leq j \leq n\,\}, \quad \sum_{j=1}^{n} |x_j|, \quad \text{and} \quad \left\{\sum_{j=1}^{n} |x_j|^2\right\}^{\frac{1}{2}},$$

respectively, for every $(x_1, \cdots, x_n) \in \mathbf{R}^n$.

The following proposition gives the matrix norms induced from these common vector norms in $\mathbf{R}^n$. Its proof can be found in [1].

Proposition A.1.2. *The matrix norm of a $n \times n$ matrix $A = (a_{ij})_{n\times n}$ induced from the l_∞, l_1, and l_2 norm in $\mathbf{R}^n$ are given by:*

$$\max\left\{\sum_{j=1}^{n} |a_{ij}| \;:\; 1 \le i \le n\right\}, \tag{1.4}$$

$$\max\left\{\sum_{i=1}^{n} |a_{ij}| \;:\; 1 \le j \le n\right\}, \tag{1.5}$$

$$\{\lambda_{\max}(A^* A)\}^{\frac{1}{2}}, \tag{1.6}$$

respectively, where $\lambda_{\max}(B)$ is the largest eigenvalue of a symmetric matrix B, and A^* is the transpose of A.

Remark. Suppose $|\cdot|_1$ and $|\cdot|_2$ are vector norms in $\mathbf{R}^n$ and $\mathbf{R}^m$, respectively. Then there is an induced matrix norm on $\mathbf{R}^{m\times n}$, the space of all $m\times n$ matrices: linear operators from $\mathbf{R}^n$ to $\mathbf{R}^m$, defined by

$$|A|_{12} \;=\; \sup_{\substack{|x|_1\neq 0\\ x\in\mathbf{R}^n}} \frac{|Ax|_2}{|x|_1} \;=\; \sup_{\substack{|x|_1\neq 0\\ x\in\mathbf{R}^n}} |Ax|_2. \tag{1.7}$$

It is easy to check that $|A|_{12}$ also satisfies the submultiplicative property, assume that $A : (\mathbf{R}^m, |\cdot|_2) \to (\mathbf{R}^k, |\cdot|_3)$ and $B : (\mathbf{R}^n, |\cdot|_1) \to (\mathbf{R}^m, |\cdot|_2)$ are two linear operators (matrices), then

$$|AB|_{13} \;\le\; |A|_{23}\, |B|_{12}.$$

§A.2. The Lozinskiĭ Measure

Let $|\cdot|$ denote a vector norm in $\mathbf{R}^n$ and the matrix norm it induces in $\mathbf{R}^{n\times n}$. The *Lozinskiĭ measure* corresponding to $|\cdot|$ is defined by

$$\mu(A) \;=\; \lim_{h\to 0+} \frac{|I + hA| - 1}{h}, \tag{2.1}$$

where $I = I_{n\times n}$ is the $n \times n$ identity matrix, or equivalently

$$\mu(A) = D_+|I + hA|_{h=0}. \tag{2.2}$$

From the convexity of the matrix norm $|\cdot|$ we know that $\mu(A)$ in (2.1) and (2.2) are well-defined. The following properties of μ are established in [1].

Proposition A.2.1. *μ has the following properties:*

(1) $-|A| \leq -\mu(-A) \leq \mu(A) \leq |A|$.

(2) $\mu(\alpha A) = \alpha\, \mu(A)$, *for all* $\alpha \geq 0$.

(3) $\max\{\mu(A) - \mu(-B), -\mu(-A) + \mu(B)\} \leq \mu(A+B) \leq \mu(A) + \mu(B)$.

(4) $\mu(A)$ *is a convex function of* A.

(5) $-\mu(-A) \leq \mathrm{Re}\lambda \leq \mu(A)$, *where* λ *is any eigenvalue of* A.

Let $|\cdot|$ be a vector norm in $\mathbf{R}^n$ and μ be the corresponding Lozinskiĭ measure. If P is a $n \times n$ nonsingular matrix, a new vector norm can be defined in $\mathbf{R}^n$ by

$$|x|_P = |Px|.$$

It is easy to check that $|\cdot|_P$ is a vector norm. Let μ_P denote the Lozinskiĭ measure corresponding to $|\cdot|_P$.

Proposition A.2.2. *For any $n \times n$ matrix A,*

$$\mu_P(A) = \mu(PAP^{-1}). \tag{2.3}$$

Proof. The matrix norm induced from $|\cdot|_P$ is given by

$$\begin{aligned}
|A|_P &= \sup_{|x|_P=1} |Ax|_P = \sup_{|x|_P=1} |PAx| \\
&= \sup_{|x|_P=1} |PAP^{-1}(Px)| = \sup_{|y|=1} |PAP^{-1}y| \\
&= |PAP^{-1}|.
\end{aligned}$$

Thus

$$\begin{aligned}\mu_P(A) &= \lim_{h\to 0+} \frac{|I+hA|_P - 1}{h} \\ &= \lim_{h\to 0+} \frac{|I+hPAP^{-1}| - 1}{h} \\ &= \mu(PAP^{-1}).\end{aligned}$$

□

The following result gives the Lozinskiĭ measures corresponding to some common vector norms in $\mathbf{R}^n$, a proof of which can be found in [1].

Proposition A.2.3. *For any matrix $A = (a_{ij})_{n\times n}$, the Lozinskiĭ measure $\mu(A)$ corresponding to the l_∞, l_1, and l_2 norms are given by*

$$\max_{1\le i\le n}\left\{ a_{ii} + \sum_{j\neq i}^{n} |a_{ij}| \right\}, \tag{2.4}$$

$$\max_{1\le j\le n}\left\{ a_{jj} + \sum_{i\neq j}^{n} |a_{ij}| \right\}, \tag{2.5}$$

$$\lambda_{\max}\Big(\frac{1}{2}(A^* + A)\Big), \tag{2.6}$$

respectively.

Remark. If the entries of A are allowed to be complex numbers, then the term a_{ii} in (2.4) and (2.5) should be replaced by $\mathrm{Re}\,a_{ii}$, the real part of a_{ii}.

Example. Consider the matrix $A = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$.
Direct calculation yields that A has two real eigenvalues 2, 1 which are contained in the interval $[1, 2]$. Using Proposition A.2.3, the Lozinskiĭ measures $\mu(A)$ and $\mu(-A)$ calculated corresponding to the l_∞, l_1, and l_2 norm of $\mathbf{R}^2$ are

$$\begin{aligned}\mu(A) &= \quad 3, \quad 2, \quad \text{and} \quad \frac{3+\sqrt{2}}{2}, \\ \mu(-A) &= -1, \quad 0, \quad \text{and} \quad \frac{-3+\sqrt{2}}{2},\end{aligned}$$

respectively. From Proposition A.2.1 we know that the real parts of all eigenvalues of A are contained in the interval $[-\mu(-A), \mu(A)]$. These intervals corresponding to the l_∞, l_1, and l_2 norm are given by

$$[0, 2], \quad [\frac{3}{2} - \frac{1}{\sqrt{2}}, \frac{3}{2} + \frac{1}{\sqrt{2}}], \quad \text{and} \quad [1, 3],$$

respectively.

We observe that in this example that the l_1 norm yields the best upper estimate of the three; the l_∞ norm gives the best lower estimate; the l_2 norm produces the interval of the smallest length.

Remark. As we have observed from the previous example, when Proposition A.2.1 is used to estimate the real parts of the eigenvalues of a matrix A, it is practical to find the interval $[-\mu(-A), \mu(A)]$ corresponding to several norms in $\mathbf{R}^n$. The real parts of the eigenvalues of A are then contained in the intersection of these intervals. In the previous example, the intersection of the three intervals we have calculated is $[1, 2]$, which is the smallest possible. Another observation we may make from the example is that the Lozinskiĭ measures corresponding to the l_∞ and l_1 norms are relatively easy to compute compared to the Lozinskiĭ measure corresponding to the l_2 norm. Allowing us to choose the appropriate vector norms to work with is one of the major advantages of the Lozinskiĭ measure.

Next we shall provide a method for the estimation of $\mu(A)$ corresponding to some more technically constructed norms. A more detailed treatment can be found in [3] and [4].

Let V_1 and V_2 be two subspaces of $\mathbf{R}^n$ with $\dim V_1 = r$, $\dim V_2 = s$, $r + s = n$, and

$$\mathbf{R}^n = V_1 \oplus V_2.$$

Then any $n \times n$ matrix A can be divided into blocks

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

and A_{ij} can be regarded as a linear operator from V_j to V_i, $i, j = 1, 2$.

Let $|\cdot|_1$ and $|\cdot|_2$ be the vector norms in V_1 and V_2, μ_1 and μ_2 the corresponding Lozinskiĭ measures, respectively. Define two vector norms in $\mathbf{R}^n$ by

$$|x| = |x_1|_1 + |x_2|_2 \tag{2.7}$$

and

$$|x| = \max\{|x_1|_1, |x_2|_2\} \tag{2.8}$$

where $x = x_1 + x_2$ with $x_i \in V_i$, $i = 1, 2$.

The Lozinskiĭ measures of A corresponding to the vector norms defined in (2.7) and (2.8) are estimated in the following proposition, a proof of which can be found in [4].

Proposition A.2.4. *An upper bound for $\mu(A)$ corresponding to the vector norms defined in (2.7) and (2.8) is given by*

$$\max\{\mu_1(A_{11}) + |A_{21}|_{12}, \quad \mu_2(A_{22}) + |A_{12}|_{21}\} \tag{2.9}$$

and

$$\max\{\mu_1(A_{11}) + |A_{12}|_{21}, \quad \mu_2(A_{22}) + |A_{21}|_{12}\}, \tag{2.10}$$

respectively.

§A.3. Asymptotic Growth of Solutions to Linear Differential Systems

Consider a linear system of ordinary differential equations in $\mathbf{R}^n$

$$x'(t) = A(t)\,x(t) \tag{3.1}$$

where $t \mapsto A(t)$ is a $n \times n$ matrix-valued function continuous in $\mathbf{R}$. Let $|\cdot|$ denote a vector norm in $\mathbf{R}^n$ and the matrix norm it induces on $\mathbf{R}^{n\times n}$. Let μ be the Lozinskiĭ measure corresponding to $|\cdot|$. We want to estimate the exponential growth and decay rate of $|x(t)|$ when $x(t)$ is a solution to (3.1).

Theorem A.3.1. *Suppose* $x = x(t)$ *is a solution to* (3.1). *Then*

$$|x(t)| \exp\left\{ -\int_s^t \mu(A(\tau))\, d\tau \right\}, \quad \text{and} \quad |x(t)| \exp\left\{ \int_s^t \mu(-A(\tau))\, d\tau \right\}$$

are decreasing and increasing in t, *respectively.*

Proof. We can write, for any $t \in \mathbf{R}$,

$$x(t+h) \;=\; x(t) \,+\, h\, x'(t) \,+\, o(h),$$

where

$$\lim_{h \to 0} \frac{|o(h)|}{h} \,=\, 0.$$

Thus

$$\begin{aligned} x(t+h) \;&=\; x(t) \,+\, h\, A(t)\, x(t) \,+\, o(h), \\ &=\; (I \,+\, h\, A(t))\, x(t) \,+\, o(h), \end{aligned}$$

and this implies

$$|x(t+h)| \;\le\; |I \,+\, h\, A(t)|\, |x(t)| \,+\, |o(h)|,$$

and

$$|x(t+h)| \,-\, |x(t)| \;\le\; \{\, |I \,+\, h\, A(t)| \,-\, 1 \,\}\, |x(t)| \,+\, |o(h)|.$$

Suppose $h > 0$, then by definition

$$\begin{aligned} D_t^+ |x(t)| \;&=\; \lim_{h \to 0+} \frac{|x(t+h)| \,-\, |x(t)|}{h} \\ &\le\; \lim_{h \to 0+} \frac{\{|I \,+\, h\, A(t)| \,-\, 1\}\, |x(t)| \,+\, |o(h)|}{h} \\ &=\; \mu\left(A(t)\right) |x(t)|, \end{aligned}$$

which implies that

$$D_t^+ \left[|x(t)| \exp\left\{ -\int_s^t \mu(A(\tau))\, d\tau \right\} \right] \le\; 0,$$

and thus $|x(t)| \exp\{-\int_s^t \mu(A(\tau))\, d\tau\}$ decreases in t as claimed. We can prove the statement on $|x(t)| \exp\{\int_s^t \mu(-A(\tau))\, d\tau\}$ similarly by considering $h < 0$. □

Theorem A.3.1 yields the following estimates on the exponential growth and decay rate of $|x(t)|$.

Corollary A.3.2. *Suppose $x(t)$ is a solution to (3.1). Then*

$$|x(s)| \exp\left\{ -\int_s^t \mu(-A(\tau))\, d\tau \right\} \le |x(t)| \le |x(s)| \exp\left\{ \int_s^t \mu(A(\tau))\, d\tau \right\} \tag{3.2}$$

for all $s \le t$.

When $|\cdot|$ is the l_2 norm, we know from §A.1 that

$$\mu(A(t)) = \lambda_1(t) \quad \text{and} \quad \mu(-A(t)) = -\lambda_n(t)$$

where $\lambda_1(t) \ge \cdots \ge \lambda_n(t)$ are eigenvalues of the symmetric matrix $(A^*(t) + A(t))/2$. This leads to the following well-known result.

Corollary A.3.3. *Suppose $\|\cdot\|$ is the l_2 norm in $\mathbf{R}^n$ and $x(t)$ is a solution to (3.1). Then*

$$\|x(s)\| \exp \int_s^t \lambda_n(\tau)\, d\tau \le \|x(t)\| \le \|x(s)\| \exp \int_s^t \lambda_1(\tau)\, d\tau \tag{3.3}$$

for all $s \le t$.

The following result follows directly from Theorem A.3.1 or Corollary A.3.2.

Corollary A.3.4. *In order for all solutions of (3.1) to satisfy*

$$\lim_{t\to\infty} |x(t)| = 0,$$

it is sufficient that

$$\lim_{t\to\infty} \int_s^t \mu(A(\tau))\, d\tau = -\infty \tag{3.4}$$

and necessary that

$$\lim_{t\to\infty} \int_s^t \mu(-A(\tau))\, d\tau \;=\; \infty. \tag{3.5}$$

§A.4. Applications to the Stability of Linear Differential Systems

The linear system (3.1) is said to be ***stable*** (***uniformly stable, asymptotically stable, uniformly asymptotically stable***) if the zero solution of (3.1) is stable (uniformly stable, asymptotically stable, uniformly asymptotically stable) (see [2]).

Given in the following are necessary and sufficient conditions for the various stabilities of (3.1). Their proofs can be found in [2].

Let $X(t)$ be a fundamental matrix of (3.1), and $\beta \in \mathbf{R}$.

Theorem A.4.1. *The linear system* (3.1) *is*

(i) *stable for any* $t_0 \in \mathbf{R}$ *if and only if there is a* $K = K(t_0) > 0$ *such that*

$$|X(t)| \;\le\; K, \qquad t_0 \le t; \tag{4.1}$$

(ii) *uniformly stable for* $t_0 \ge \beta$ *if and only if there exists a* $K = K(\beta) > 0$ *such that*

$$|X(t)X^{-1}(s)| \;\le\; K, \qquad t_0 \le s \le t; \tag{4.2}$$

(iii) *asymptotically stable for any* $t_0 \in \mathbf{R}$ *if and only if*

$$\lim_{t\to\infty} |X(t)| \;=\; 0; \tag{4.3}$$

(iv) *uniformly asymptotically stable for* $t_0 \ge \beta$ *if and only if it is exponentially stable; that is, there are* $K = K(\beta) > 0$, *and* $\alpha = \alpha(\beta) > 0$ *such that*

$$|X(t)X^{-1}(s)| \;\le\; K\exp\{-\alpha(t-s)\}, \qquad t_0 \le s \le t. \tag{4.4}$$

The following concrete criteria for various stabilities of (3.1) are derived from Theorem A.4.1 and Theorem A.3.1. Detailed proofs can be found in [1].

Theorem A.4.2. *The linear system* (3.1) *is*

(i) *stable for any* $t_0 \in \mathbf{R}$ *if*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \mu\big(A(s)\big)\, ds \;<\; \infty; \tag{4.5}$$

(ii) *uniformly stable for* $t_0 \geq \beta$ *if there exists a* $M > 0$, *such that*

$$\int_{s}^{t} \mu\big(A(\tau)\big)\, d\tau \;\leq\; M \;<\; \infty, \qquad t_0 \leq s \leq t; \tag{4.6}$$

(iii) *asymptotically stable for any* $t_0 \in \mathbf{R}$ *if*

$$\lim_{t\to\infty} \int_{t_0}^{t} \mu\big(A(s)\big)\, ds \;=\; -\infty; \tag{4.7}$$

(iv) *uniformly asymptotically stable for* $t_0 \geq \beta$ *if there exists a* $\lambda = \lambda(\beta)$ *such that*

$$\mu\big(A(t)\big) \;\leq\; -\lambda \;<\; 0, \qquad t_0 \leq t. \tag{4.8}$$

Remarks.

(i). It is well known that when $A(t)$ in (3.1) is a constant matrix A, the stability character of (3.1) is determined by the real parts of the eigenvalues of A. This is no longer the case, however, if $A(t)$ is not constant: the system (3.1) may not be asymptotically stable even if the real parts of all eigenvalues of $A(t)$ are negative for all t. In this case, the criteria given in Theorem A.4.2 may be very useful to determine the stability of (3.1).

(ii). For applications of the Lozinskiĭ measure in the study of dichotomy of (3.1), readers are referred to [4].

§A.5. Bibliography for Apendix A

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APPENDIX B

COMPOUND MATRICES AND COMPOUND EQUATIONS

In this appendix, we present a theory of compound matrices and compound equations which is extensively used in the thesis.

Most of the development on compound matrices have occurred in the context of linear and multilinear algebra. There is an extensive classical body of work dealing with algebraic aspects of compound matrices (see [2] [5] [7]). Good historical accounts may be found in [8] and [12], or in [9] and its references.

Applications of compound matrices to differential equations began in the 1970's. The first treatment in full generality is due to Schwarz [13] in his study on the total positivity of fundamental matrices to general linear systems. In his series of papers [9] [10] [11], Muldowney systematically uses the theory of compound equations in his study of some important linear and nonlinear problems of differential equations such as dichotomy and stability theory, orbital stability of periodic solutions of nonlinear autonomous systems and higher dimensional generalizations of Bendixson's criterion.

Here we present a treatment of the theory of compound equations based on multilinear algebra. It has the advantage of producing a more concise and easy-to-follow presentation, even though familiarity with tensor algebra is required. Another advantage of this treatment is that it facilitates expansion and further generalization of the theory.

In Section B.1, fundamentals of tensor algebra are reviewed; in Section B.2, compound matrices are defined as matrix representations for certain linear operators on the exterior product space of $\mathbf{R}^n$. Their calculations along with their algebraic and spectral properties are also discussed; in Section B.3, compound equations are introduced as equations describing evolution of volume elements of various dimensions in $\mathbf{R}^n$.

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§B.1. Exterior Products in $\mathbf{R}^n$

We denote the euclidean inner product and norm by $(,)$ and $\|\cdot\|$, respectively. We also let $\{e_j\}_{j=1}^n$ denote the standard orthonormal basis in $\mathbf{R}^n$.

For any integer $1 \le k \le n$, we wish to define a vector space $\bigotimes^k \mathbf{R}^n$: *the k-th tensor product of* $\mathbf{R}^n$. We do so by describing its elements. Let $u_1, \cdots, u_k$ be k elements of $\mathbf{R}^n$. Their *tensor product* $u_1 \otimes \cdots \otimes u_k$ is a k-linear functional on $\mathbf{R}^n$ defined by

$$u_1 \otimes \cdots \otimes u_k (v_1, \cdots, v_k) = \prod_{j=1}^{k} (u_j, v_j) \tag{1.1}$$

for all $v_1, \cdots, v_k \in \mathbf{R}^n$. The vector space $\bigotimes^k \mathbf{R}^n$ is the linear span of all elements of form $u_1 \otimes \cdots \otimes u_k$, $u_1, \cdots, u_k \in \mathbf{R}^n$. It is not hard to see that the set

$$\{ e_{i_1} \otimes \cdots \otimes e_{i_k} \ : \ 1 \le i_1, \cdots, i_k \le n \} \tag{1.2}$$

is a basis of $\bigotimes^k \mathbf{R}^n$ and thus the dimension of $\bigotimes^k \mathbf{R}^n$ is n^k.

An inner product $(\,,\,)_k$ can be defined on $\bigotimes^k \mathbf{R}^n$ canonically from that of $\mathbf{R}^n$. For two elements $u_1 \otimes \cdots \otimes u_k$ and $v_1 \otimes \cdots \otimes v_k$ of $\bigotimes^k \mathbf{R}^n$, we define

$$(u_1 \otimes \cdots \otimes u_k, \ v_1 \otimes \cdots \otimes v_k)_k = \prod_{j=1}^{k} (u_j, v_j) \tag{1.3}$$

and extend this definition to all of $\bigotimes^k \mathbf{R}^n$ using the bi-linearity of the inner product. We denote the norm derived from $(\,,\,)_k$ by $\|\cdot\|_k$ i.e.

$$\|\Phi\|_k = \{(\Phi, \, \Phi)_k\}^{1/2} \tag{1.4}$$

for all $\Phi \in \bigotimes^k \mathbf{R}^n$. Obviously when $k = 1$, $(\,,\,)_k$ and $\|\cdot\|_k$ coincide with $(\,,\,)$ and $\|\cdot\|$, respectively.

Under $(\,,\,)_k$ and $\|\cdot\|_k$, the basis of $\bigotimes^k \mathbf{R}^n$ given in (1.2) is orthonormal since

$$\begin{aligned} &(e_{i_1} \otimes \cdots \otimes e_{i_k}, \, e_{j_1} \otimes \cdots \otimes e_{j_k})_k \\ &\qquad = \prod_{s=1}^{k} (e_{i_s}, e_{j_s}) \, = \, \prod_{s=1}^{k} \delta_{j_s}^{i_s} \, =: \, \delta_{j_1 \cdots j_k}^{i_1 \cdots i_k}, \end{aligned}$$

where δ^i_j is the Kronecker symbol, i.e.

$$\delta^i_j = \begin{cases} 0, & i \neq j, \\ 1, & i = j. \end{cases}$$

We can see from its dimension that $\bigotimes^k \mathbf{R}^n$ is a huge space. What is more interesting to us in this thesis is the subspace of $\bigotimes^k \mathbf{R}^n$ consisting of anti-symmetric k-linear functionals on $\mathbf{R}^n$, the so-called *k-exterior products* or *k-wedge products.*

A permutation on the set of integers $\{1, \cdots, k\}$ is a bijective map

$$\sigma : \{1, \cdots, k\} \to \{1, \cdots, k\}.$$

Such a σ includes a linear operator on $\bigotimes^k \mathbf{R}^n$ in the following manner: if Φ is an element of $\bigotimes^k \mathbf{R}^n$, i.e., a k-linear functional on $\mathbf{R}^n$, a new element $\sigma\Phi \in \bigotimes^k \mathbf{R}^n$ is defined by its action on $\mathbf{R}^n$:

$$\sigma\Phi\,(v_1, \cdots, v_k) =: \Phi\big(v_{\sigma(1)}, \cdots, v_{\sigma(k)}\big)$$

for all $v_1, \cdots, v_k \in \mathbf{R}^n$. In particular, if $\Phi = u_1 \otimes \cdots \otimes u_k$, then

$$\sigma\big(u_1 \otimes \cdots \otimes u_k\big) = u_{\sigma^{-1}(1)} \otimes \cdots \otimes u_{\sigma^{-1}(k)}$$

where σ^{-1} is the inverse mapping of σ. The product of two permutations σ and τ is defined by the composition of mappings $\sigma \circ \tau$. Under this product all permutations on $\{1, \cdots, k\}$ form a multiplicative group $\mathbf{S}_k$ whose index is $k\,!$.

An element $\Phi \in \bigotimes^k \mathbf{R}^n$ is said to be *symmetric* if $\sigma\Phi = \Phi$ for all $\sigma \in \mathbf{S}_k$; *anti-symmetric* if $\sigma\Phi = \mathrm{sgn}(\sigma)\,\Phi$, where sgn (σ) is such that

$$\mathrm{sgn}(\sigma) = \begin{cases} 1, & \text{if } \sigma \text{ is even}, \\ -1, & \text{if } \sigma \text{ is odd}. \end{cases}$$

All anti-symmetric elements in $\bigotimes^k \mathbf{R}^n$ form a subspace $\bigwedge^k \mathbf{R}^n$. In the rest of the section, we want to describe elements of this subspace.

The *alternation mapping* $\mathcal{A}_k : \bigotimes^k \mathbf{R}^n \to \bigotimes^k \mathbf{R}^n$ is defined by

$$\mathcal{A}_k\,\Phi = \frac{1}{k!} \sum_{\sigma \in \mathbf{S}_k} \mathrm{sgn}(\sigma)\,\sigma\Phi \tag{1.5}$$

for all $\Phi \in \bigotimes^k \mathbf{R}^n$. We have the following result, a proof of which can be found in [1].

Proposition B.1.1. *The mapping* $\mathcal{A}_k$ *satisfies:*

(1) $\mathcal{A}_k$ *is a homomorphism.*

(2) $\bigwedge^k \mathbf{R}^n = \mathcal{A}_k(\bigotimes^k \mathbf{R}^n)$.

(3) $\mathcal{A}_k|_{\bigwedge^k \mathbf{R}^n} = \mathrm{id}$.

(4) $\mathcal{A}_k \circ \mathcal{A}_k = \mathcal{A}_k$.

For $u_1, \cdots, u_k \in \mathbf{R}^n$, the element

$$u_1 \wedge \cdots \wedge u_k =: \mathcal{A}_k(u_1 \otimes \cdots \otimes u_k) \tag{1.6}$$

belongs to $\bigwedge^k \mathbf{R}^n$ by Proposition B.1.1. It is called the *exterior product* of $u_1, \cdots, u_k$. We know from the anti-symmetric property satisfied by the elements in $\bigwedge^k \mathbf{R}^n$ that $u_1 \wedge \cdots \wedge u_k = 0$ if $u_i = u_j$ for some $i \neq j$. We can also deduce from Proposition B.1.1 that

$$\{ e_{i_1} \wedge \cdots \wedge e_{i_k} \ : \ 1 \leq i_1 < \cdots < i_k \leq n \} \tag{1.7}$$

is a basis for $\bigwedge^k \mathbf{R}^n$, since it is the image under the linear mapping $\mathcal{A}_k$ of the basis of $\bigotimes^k \mathbf{R}^n$ given in (1.2). In particular, the dimension of $\bigwedge^k \mathbf{R}^n$ is $\binom{n}{k}$.

Remark. It will be seen later that the order of elements in the basis given in (1.7) is very important. Throughout this thesis we always assume that they are ordered lexicographically.

As a subspace of $\bigotimes^k \mathbf{R}^n$, $\bigwedge^k \mathbf{R}^n$ inherits an inner product, we also denote it by $(\ ,\ )_k$ and the corresponding norm by $\|\cdot\|_k$.

Proposition B.1.2. *Suppose* $u_1, \cdots, u_k$, $v_1, \cdots, v_k \in \mathbf{R}^n$. *Then*

$$(u_1 \wedge \cdots \wedge u_k,\ v_1 \wedge \cdots \wedge v_k)_k = \det\big((u_i, v_j)\big)_{1 \leq i,j \leq n}. \tag{1.8}$$

Proof. By the definition of exterior products and (1.3), we have

$$
\begin{aligned}
&\big(u_1 \wedge \cdots \wedge u_k,\ v_1 \wedge \cdots \wedge v_k\big)_k \\
&= \big(\mathcal{A}_k(u_1 \otimes \cdots \otimes u_k),\ \mathcal{A}_k(v_1 \otimes \cdots \otimes v_k)\big)_k \\
&= \sum_{\sigma \in \mathbf{S}_k} \sum_{\tau \in \mathbf{S}_k} \big(\operatorname{sgn}(\sigma)\, u_{\sigma^{-1}(1)} \otimes \cdots \otimes u_{\sigma^{-1}(k)},\ \operatorname{sgn}(\tau)\, v_{\tau^{-1}(1)} \otimes \cdots \otimes v_{\tau^{-1}(k)}\big)_k \\
&= \sum_{\sigma \in \mathbf{S}_k} \sum_{\tau \in \mathbf{S}_k} \operatorname{sgn}(\sigma)\ \operatorname{sgn}(\tau) \prod_{i=1}^{k} \big(u_{\sigma^{-1}(i)},\ v_{\tau^{-1}(i)}\big) \\
&= \sum_{\sigma \in \mathbf{S}_k} \operatorname{sgn}(\sigma) \prod_{i=1}^{k} \big(u_i,\ v_{\sigma^{-1}(i)}\big) \\
&= \det\big((u_i, v_j)\big)_{1 \le i,j \le n}.
\end{aligned}
$$

□

It follows from Proposition B.1.2 that the basis of $\bigwedge^k \mathbf{R}^n$ given in (1.7) is orthonormal. We want to see how elements of $\bigwedge^k \mathbf{R}^n$ can be expressed in coordinates relative to this basis. Let $u_1, \cdots, u_k \in \mathbf{R}^n$ and $u_i = \sum_{j=1}^n a_{ij}\, e_j,\ i = 1, \cdots, k$. Then

$$
\begin{aligned}
u_1 \wedge \cdots \wedge u_k &= \Big(\sum_{j=1}^{n} a_{ij}\, e_j\Big) \wedge \cdots \wedge \Big(\sum_{j=1}^{n} a_{kj}\, e_j\Big) \\
&= \sum_{i_1 < \cdots < i_k} \kappa_{i_1 \cdots i_k}\, e_{i_1} \wedge \cdots \wedge e_{i_k},
\end{aligned}
\tag{1.9}
$$

where the sum is over all possible k-tuples of integers $(i_1, \cdots, i_k)$ such that $1 \le i_1 < \cdots < i_k \le n$ and $\kappa_{i_1 \cdots i_k}$ is the determinant of $k \times k$ block of the matrix $(a_{ij})_{\substack{1 \le i \le k \\ 1 \le j \le n}}$ formed by elements in the i_1-th, $\cdots$, and i_k-th row.

Example. For two vectors $u = (1, 0, 2)$ and $v = (0, 2, 1)$ in $\mathbf{R}^3$, $u \wedge v$ is a vector in $\mathbf{R}^{\binom{3}{2}} \cong \mathbf{R}^3$. Moreover,

$$
\begin{aligned}
u \wedge v &= \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} e_1 \wedge e_2 + \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} e_1 \wedge e_3 + \begin{vmatrix} 0 & 2 \\ 2 & 1 \end{vmatrix} e_2 \wedge e_3 \\
&= 2\, e_1 \wedge e_2 + e_1 \wedge e_3 - 4\, e_2 \wedge e_3.
\end{aligned}
$$

In the special case when $k = n$, we have

$$u_1 \wedge \cdots \wedge u_n = \left|\det\left(a_{ij}\right)_{1 \leq i,j \leq n}\right| e_1 \wedge \cdots \wedge e_n, \tag{1.10}$$

which gives rise to the following important property.

Proposition B.1.3. *Elements* $u_1, \cdots, u_k$ *of* $\mathbf{R}^n$ *are linearly dependent if and only if* $u_1 \wedge \cdots \wedge u_k = 0$.

Proof. When $k = n$ this follows directly from (1.10). If $k < n$, we may choose an orthonormal basis $\{v_1, \cdots, v_k\}$ of the subspace of $\mathbf{R}^n$ spanned by $\{u_1, \cdots, u_k\}$, and then use (1.10) to prove the proposition. □

What is most interesting to us here is the connection of exterior products with volume elements in $\mathbf{R}^n$.

Suppose $u_1, \cdots, u_k$ are linearly independent vectors in $\mathbf{R}^n$. We denote by $\mathcal{K}$ the k-dimensional parallelepiped (see Figure B.1.1) in $\mathbf{R}^n$, i.e.

$$\mathcal{K} = \left\{ \sum_{i=1}^{k} \lambda_i u_i \ : \ 0 \leq \lambda_i \leq 1, \ \sum_{i=1}^{k} \lambda_i = 1 \right\}.$$

In order to calculate the k-dimensional volume of $\mathcal{K}$, denoted by $\mathrm{Vol}_k(\mathcal{K})$, we choose an orthonormal basis $\{v_1, \cdots, v_k\}$ of the subspace of $\mathbf{R}^n$ spanned by $u_1, \cdots, u_k$ and write

$$u_i = \sum_{j=1}^{k} a_{ij} v_j, \qquad i = 1, \cdots, k.$$

Then

$$\mathrm{Vol}_k(\mathcal{K}) = \left|\det(a_{ij})_{1 \leq i,j \leq k}\right|.$$

On the other hand from (1.10)

$$u_1 \wedge \cdots \wedge u_k = \left|\det(a_{ij})\right| v_1 \wedge \cdots \wedge v_k.$$

Since $\|v_1 \wedge \cdots \wedge v_k\|_k = 1$, we arrive at the following result.

Proposition B.1.4.

$$\mathrm{Vol}_k(\mathcal{K}) = \|u_1 \wedge \cdots \wedge u_k\|_k. \tag{1.11}$$

Example. For two vectors $u, v \in \mathbf{R}^n$,

$$\begin{aligned}
\|u \wedge v\| &= \left|\det \begin{bmatrix} (u,u) & (u,v) \\ (v,u) & (v,v) \end{bmatrix}\right|^{\frac{1}{2}} \\
&= \|u\|\,\|v\|\,(1 - \frac{(u,\,v)^2}{\|u\|^2\,\|v\|^2})^{\frac{1}{2}} \\
&= \|u\|\,\|v\|\,\sin\theta \\
&= \text{Area of the parallelogram spanned by } u \text{ and } v,
\end{aligned}$$

where θ is the angle between u and v given by

$$\cos\theta = \frac{(u,\,v)}{\|u\|\,\|v\|}.$$

(See Figure B.1.2).

We summarize the above discussion in the following theorem.

Theorem B.1.5.

(1) $u_1 \wedge \cdots \wedge u_k = 0 \Leftrightarrow u_1, \cdots, u_k$ *are linearly dependent in* $\mathbf{R}^n$.

(2) $\{e_{i_1} \wedge \cdots \wedge e_{i_k} \;:\; 1 \le i_1 < \cdots < i_k \le n\}$ *is an orthonormal basis for* $\bigwedge^k \mathbf{R}^n$.

(3) *If* $\mathcal{K} = \mathcal{K}(u_1, \cdots, u_k) = \{\sum_{i=1}^k \lambda_i u_i \;:\; \sum_{i=1}^i \lambda_i = 1,\ 0 \le \lambda_1, \cdots, \lambda_k \le 1\}$,

then

$$\mathrm{Vol}_k(\mathcal{K}) = \|u_1 \wedge \cdots \wedge u_k\|_k.$$

(4) $(u_1 \wedge \cdots \wedge u_k,\, v_1 \wedge \cdots \wedge v_k)_k = \det\big((u_i, v_j)\big)_{1 \le i,j \le k}$.

(5) $\|v_1 \wedge \cdots \wedge u_k\|_k \le \|u_1\| \cdots \|u_k\|$.

Remark. Let r and s be two integers and $\Delta \in \bigwedge^r \mathbf{R}^n$ an $\Omega \in \bigwedge^s \mathbf{R}^n$. We can define more generally the exterior product $\Delta \wedge \Omega$ of Δ and Ω. It is an element

of $\bigwedge^{r+s}\mathbf{R}^n$ given by

$$\Lambda \wedge \Omega\,(v_1, \cdots, v_{r+s}) =$$
$$= \sum \operatorname{sgn}(\sigma)\, \Delta\big(v_{\sigma(1)}, \cdots, v_{\sigma(r)}\big)\, \Omega\big(v_{\sigma(r+1)}, \cdots, v_{\sigma(r+s)}\big). \tag{1.12}$$

where the sum is over all permutations σ on $\{1, \cdots, r+s\}$ such that $\sigma(1) < \cdots < \sigma(r)$ and $\sigma(r+1) < \cdots < \sigma(r+s)$. The followings are basic properties of the exterior product $\bigwedge$:

$$\wedge \quad \text{is bilinear on} \quad \bigwedge^{r}\mathbf{R}^n \times \bigwedge^{s}\mathbf{R}^n, \tag{1.13}$$

$$\Delta \wedge \Omega = (-1)^{r+s}\,\Omega \wedge \Delta, \tag{1.14}$$

$$\Delta \wedge (\Omega \wedge \Theta) = (\Delta \wedge \Omega) \wedge \Theta. \tag{1.15}$$

Let $\bigwedge^0 \mathbf{R}^n = \mathbf{R}$ and define

$$\bigwedge \mathbf{R}^n = \bigoplus_{k=1}^{\infty} \bigwedge^{k} \mathbf{R}^n.$$

Then $\bigwedge \mathbf{R}^n$ is a vector space endowed with a multiplication given by $\bigwedge$. $\bigwedge \mathbf{R}^n$ is called the *exterior algebra* of $\mathbf{R}^n$ or the *Grassman algebra* of $\mathbf{R}^n$. For more details and further studies on this subject, readers are referred to [1], [6], and [14].

§B.2. Compound Matrices and Their Properties

In this section, we define and discuss the multiplicative and additive compound matrices. In the Subsection B.2.1, two linear operators $T^{(k)}$ and $T^{[k]}$ on $\bigwedge^k \mathbf{R}^n$ are introduced. Then, in the Subsection B.2.2, compound matrices will be defined as the matrix representations of these two linear operators. Algebraic properties of compound matrices will also be discussed in this subsection. In the Subsection B.2.3, we compute the norm of compound matrices and discuss their spectral properties.

§B.2.1 Induced Linear Operators on Exterior Product Spaces

Let $T : \mathbf{R}^n \to \mathbf{R}^n$ be a bounded linear operator. We will discuss two linear operators on $\bigwedge^k \mathbf{R}^n$ induced canonically from T. For $u_1, \cdots, u_k \in \mathbf{R}^n$, define

$$T^{(k)}(u_1 \wedge \cdots \wedge u_k) = Tu_1 \wedge \cdots \wedge Tu_k \tag{2.1}$$

and

$$T^{[k]}(u_1 \wedge \cdots \wedge u_k) = \sum_{j=1}^{k} u_1 \wedge \cdots \wedge Tu_j \wedge \cdots \wedge u_k \tag{2.2}$$

and extend the definition of $T^{(k)}$ and $T^{[k]}$ linearly to all of $\bigwedge^k \mathbf{R}^n$. Then it is easy to check that both $T^{(k)}$ and $T^{[k]}$ are linear operators on $\bigwedge^k \mathbf{R}^n$. The following properties are direct results of (2.1) and (2.2).

Proposition B.2.1. *Suppose* T_1, T_2 *are linear operators on* $\mathbf{R}^n$. *Then*

(1) $(T_1 T_2)^{(k)} = T_1^{(k)} T_2^{(k)}$,

(2) $(T_1 + T_2)^{[k]} = T_1^{[k]} + T_2^{[k]}$.

Remark. $T^{(k)}$ is sometimes called the k-th exterior power of T and denoted by $\bigwedge^k T$.

We shall study $T^{(k)}$ and $T^{[k]}$ by their matrix representations with respect to the canonical basis of $\bigwedge^k \mathbf{R}^n$.

§B.2.2 Compound Matrices

Relative to the standard basis $\{e_1, \cdots, e_n\}$ of $\mathbf{R}^n$, a linear operator $T : \mathbf{R}^n \to \mathbf{R}^n$ can be represented by a matrix $A = (a_i^j)_{n\times n}$ in the following manner:

$$T\, e_i = \sum_{j=1}^{n} a_i^j\, e_j \qquad i = 1, \cdots, n.$$

Let $A^{(k)}$ and $A^{[k]}$ be the matrix representations of the linear operators $T^{(k)}$ and $T^{[k]}$ on $\bigwedge^k \mathbf{R}^n$, respectively, relative to the basis

$$\{\, e_{i_1} \wedge \cdots \wedge e_{i_k} \;:\; 1 \le i_1 < \cdots < i_k \le n \,\}$$

in the lexicographic order. Due to Proposition B.2.1, $A^{(k)}$ and $A^{[k]}$ are called the k-th *multiplicative* and the k-th *additive compound matrix* of A, respectively. Obviously both $A^{(k)}$ and $A^{[k]}$ are $N \times N$ square matrices, $N = \binom{n}{k}$. Next we calculate their entries in terms of those of A.

For any integer $i = 1, \cdots, N$, $N = \binom{n}{k}$, let $(i) = (i_1, \cdots, i_k)$ be the i-th member in the lexicographic ordering of all k-tuples of integers such that $1 \le i_1 < \cdots < i_k \le n$. For $(i) = (i_1, \cdots, i_k)$, $(j) = (j_1, \cdots, j_k)$, let $a^{(j)}_{(i)}$ denote the minor of A determined by the rows $(i_1, \cdots, i_k)$ and columns $(j_1, \cdots, j_k)$.

Proposition 2.2. *Let* $Y = A^{(k)}$. *Then, for any* $1 \le i, j \le N$, $N = \binom{n}{k}$, *the entry* y^j_i *of* Y *is given by*

$$y^j_i = a^{(j)}_{(i)}. \tag{2.3}$$

Proof. By definition

$$\begin{aligned} y^j_i &= \left(T^{(k)}(e_{i_1} \wedge \cdots \wedge e_{i_k}),\, e_{j_1} \wedge \cdots \wedge e_{j_k}\right)_k \\ &= \left(T\,e_{i_1} \wedge \cdots \wedge Te_{i_k},\, e_{j_1} \wedge \cdots \wedge e_{j_k}\right)_k \\ &= \det\left(T\,e_{i_s},\, e_{j_t}\right)_{1 \le s,t \le k} \\ &= \det\left(a^{j_t}_{i_s}\right)_{1 \le s,t \le k} = a^{(j)}_{(i)}. \end{aligned}$$

□

Remark. The multiplicative compound matrix $A^{(k)}$ can be defined even when A is not a square matrix. Readers are referred to [7] and [9] for details.

Proposition B.2.3. *Let* $Z = A^{[k]}$. *Then, for any* $1 \le i, j \le N$, $N = \binom{n}{k}$, *the entry* z^j_i *of* Z *is given by*

$$z^j_i = \begin{cases} a^{i_1}_{i_1} + \cdots + a^{i_k}_{i_k}, & \text{if } (i) = (j) \\ (-1)^{r+s} a^{j_r}_{i_s}, & \text{if exactly one entry } i_s \text{ of } (i) \text{ does not} \\ & \text{occur in } (j) \text{ and } j_r \text{ does not occur in } (i), \\ 0, & \text{if } (i) \text{ differs from } (j) \text{ in two or more entries.} \end{cases}$$

Proof. By definition

$$
\begin{aligned}
z_i^j &= \left(T^{[k]} e_{i_1} \wedge \cdots \wedge e_{i_k}, e_{j_1} \wedge \cdots \wedge e_{j_k}\right)_k \\
&= \sum_{s=1}^{k} \left(e_{i_1} \wedge \cdots \wedge T e_{i_s} \wedge \cdots \wedge e_{i_k}, e_{j_1} \wedge \cdots \wedge e_{j_k}\right)_k \\
&= \sum_{s=1}^{k} \begin{vmatrix} (e_{i_1}, e_{j_1}) & \cdots & (e_{i_1}, e_{j_k}) \\ \vdots & \ddots & \vdots \\ (Te_{i_s}, e_{j_1}) & \cdots & (Te_{i_s}, e_{j_k}) \\ \vdots & \ddots & \vdots \\ (e_{i_k}, e_{j_1}) & \cdots & (e_{i_k}, e_{j_k}) \end{vmatrix} \\
&= \sum_{s=1}^{k} \begin{vmatrix} \delta_{i_1}^{j_1} & \cdots & \delta_{i_1}^{j_k} \\ \vdots & \ddots & \vdots \\ a_{i_s}^{j_1} & \cdots & a_{i_s}^{j_k} \\ \vdots & \ddots & \vdots \\ \delta_{i_k}^{j_1} & \cdots & \delta_{i_k}^{j_k} \end{vmatrix}.
\end{aligned}
$$

If (i) and (j) differ in two or more entries, then each determinant will have at least one row identically zero, and thus $z_i^j = 0$. In the case when $(i) = (j)$, in the s-th determinant for each s, the s-th row is $(\cdots, a_{i_s}^{i_s}, \cdots)$ and the remaining rows all have off-diagonal elements zero and diagonal elements 1. Therefore this determinant is given by $a_{i_s}^{i_s}$, and $z_i^j = a_{i_1}^{i_1} + \cdots + a_{i_k}^{i_k}$. Suppose now exactly one entry i_s of (i) does not occur in (j) and j_r does not occur in (i). Then all determinants except the s-th one have at least one row of zeroes and thus all vanish, and the i_r-th column in the s-th determinant is $\left(0, \cdots, a_{i_s}^{j_r}, \cdots 0\right)^*$. Expanding this determinant along the j_r-th column, we find its value is $(-1)^{r+s}\, a_{i_s}^{j_r}$. As the result, $z_i^j = (-1)^{r+s}\, a_{i_s}^{j_r}$. □

The following results list algebraic properties of compound matrices $A^{(k)}$ and $A^{[k]}$.

Proposition B.2.4. *Let A, B be $n \times n$ matrices. Then*

(1) $(A\,B)^{(k)} = A^{(k)}\,B^{(k)}$.

(2) *If* $I_{n\times n}$ *is the* $n \times n$ *identity matrix then* $I_{n\times n}^{(k)} = I_{N\times N}$, $N = \binom{n}{k}$.

(3) *If* A *is nonsingular, so is* $A^{(k)}$ *and* $\left(A^{(k)}\right)^{-1} = \left(A^{-1}\right)^{(k)}$.

(4) $\left(A^{(k)}\right)^* = \left(A^*\right)^{(k)}$.

(5) *If* A *is symmetric (unitary, normal), so is* $A^{(k)}$.

(6) $A^{(1)} = A$, *and* $A^{(n)} = \det A$.

(7) *If A is triangular, so is* $A^{(k)}$ *and the diagonal element of* $A^{(k)}$ *corresponding to the index* $(i) = (i_1, \cdots, i_k)$ *is* $a_{i_1 i_1} \cdots a_{i_k i_k}$.

Proof. The proofs for (1) — (6) are straightforward. To show (7) observe that if A is upper triangular, so is every block $a_{i_1 \cdots i_k}^{j_1 \cdots j_k}$ of A, and at least one of its diagonal elements is 0 if $(i) > (j)$ in the lexicographic ordering; when $(i) = (j)$, the diagonal elements of $a_{i_1 \cdots i_k}^{j_1 \cdots j_k}$ are $a_{i_1 i_1}, \cdots, a_{i_k i_k}$. Therefore (7) follows from Proposition B.2.2. □

Proposition B.2.5. *Let* A, B *be* $n \times n$ *matrices. Then*

(1) $(A + B)^{[k]} = A^{[k]} + B^{[k]}$.

(2) *If* $I_{n\times n}$ *is the* $n \times n$ *identity matrix, then* $I_{n\times n}^{[k]} = k\, I_{N\times N}$, $N = \binom{n}{k}$.

(3) $(A^*)^{[k]} = (A^{[k]})^*$.

(4) *If* A *is symmetric (anti-symmetric), so is* $A^{[k]}$.

(5) $A^{[1]} = A$, *and* $A^{[n]} = \operatorname{tr}(A)$.

(6) *If* A *is triangular, so is* $A^{[k]}$. *Moreover, the diagonal entry of* $A^{[k]}$ *corresponding to the index* $(i) = (i_1, \cdots, i_k)$ *is* $a_{i_1 i_1} + \cdots + a_{i_k i_k}$.

Proof. The proofs for (1) — (5) are straightforward and (6) follows directly from Proposition B.2.3. □

§B.2.3 Norms and Spectral Properties

The following result gives the spectral properties of $A^{(k)}$ and $A^{[k]}$.

Proposition B.2.6. *Suppose* $\lambda_1, \cdots, \lambda_n$ *are eigenvalues of a* $n \times n$ *matrix* A. *Then*

(1) *The eigenvalues of* $A^{(k)}$ *are given by all possible products of the form:*

$$\lambda_{i_1} \cdots \lambda_{i_k}, \quad 1 \le i_1 < \cdots < i_k \le n.$$

(2) *The eigenvalues of* $A^{[k]}$ *are given by all possible sums of the form:*

$$\lambda_{i_1} + \cdots + \lambda_{i_k}, \quad 1 \le i_1 < \cdots < i_k \le n.$$

(3) *Suppose* $x_1, \cdots, x_k$ *are independent eigenvectors of* A *corresponding to eigenvalues* $\lambda_{i_1}, \cdots, \lambda_{i_k}$. *Then* $x_1 \wedge \cdots \wedge x_k$ *is the eigenvector of* $A^{(k)}$ *and* $A^{[k]}$, *corresponding to the eigenvalue* $\lambda_{i_1} \cdots \lambda_{i_k}$ *and* $\lambda_{i_1} + \cdots + \lambda_{i_k}$, *respectively.*

Proof. Complexify $\mathbf{R}^n$ to obtain $\mathbf{C}^n$. Considered as a matrix over the field of complex numbers, A can be transformed into a triangular matrix B by a change of basis in $\mathbf{C}^n$; that is, there is a nonsingular complex matrix S such that

$$A = SBS^{-1}$$

and diagonal elements of B are exactly the eigenvalues of A (see [3] or [5]). Both A and B can be considered as matrix representations of the same linear operator $T : \mathbf{R}^n \to \mathbf{R}^n$ with respect to different bases of $\mathbf{R}^n$. In the same way, both $A^{(k)}$ and $B^{(k)}$ are matrix representations of $T^{(k)}$ relative to different bases of $\bigwedge^k \mathbf{R}^n$. It then follows that $A^{(k)}$ and $B^{(k)}$ have the same eigenvalues. Now from Proposition B.2.4 we know $B^{(k)}$ is triangular; its eigenvalues are given by its diagonal elements which are $\lambda_{i_1} + \cdots + \lambda_{i_k}$, $1 \le i_1 < \cdots < i_k \le n$. Thus (1) is proved. (2) can be proved using a similar argument, and (3) follows from the definition of $A^{(k)}$ and $A^{[k]}$. □

Recall that for a vector norm $|\cdot|$ in $\mathbf{R}^n$, the matrix norm of a $n \times n$ matrix A induced by $|\cdot|$ is defined as

$$|A| = \sup_{|x|=1} |Ax|.$$

In the rest of the section, we want to derive the matrix norm of $A^{(k)}$ induced by the euclidean norm $\|\cdot\|$ of $\mathbf{R}^n$.

The *singular values* of a $n \times n$ matrix B are the eigenvalues of the symmetric matrix $\sqrt{B^*B}$. We denote them by $\sigma_1 \geq \cdots \geq \sigma_n \geq 0$. The following theorem is proved in [3].

Theorem B.2.7 (The Minmax Principle).

$$\begin{aligned}\sigma_j &= \max_{\substack{V\subset\mathbf{R}^m\\ \dim V=j}} \min_{\substack{u\in V\\ \|u\|=1}} \|Bu\| \\ &= \min_{\substack{W\subset\mathbf{R}^m\\ \dim W=m-j+1}} \max_{\substack{u\in W\\ \|u\|=1}} \|Bu\|,\end{aligned}$$

where $\|\cdot\|$ *is the matrix norm induced from the euclidean norm of* $\mathbf{R}^n$.

Corollary B.2.8. $\sigma_1 = \|B\|$.

Proposition B.2.9. *For a* $n \times n$ *matrix* A, *and* $1 \leq k \leq n$,

$$\|A^{(k)}\| = \sigma_1 \cdots \sigma_k. \tag{2.7}$$

Proof. Since $\sigma_1^2, \cdots, \sigma_n^2$ are eigenvalues of A^*A, products of the form

$$\sigma_{i_1}^2 \cdots \sigma_{i_k}^2, \qquad 1 \leq i_1 < \cdots < i_k \leq n$$

are eigenvalues of $A^{(k)*}A^{(k)}$ by Proposition B.2.6. Therefore $\sigma_{i_1} \cdots \sigma_{i_k}$, $1 \leq i_1 < \cdots < i_k \leq n$ are singular values of $A^{(k)}$. Hence (2.7) follows directly from Corollary B.2.8. □

The following important relationships are established in [9].

Proposition B.2.10. *For a* $n \times n$ *matrix* A,

(1) $A^{[k]} = D(I + hA)^{(k)}\big|_{h=0}$.

(2) $(\exp A)^{(k)} = \exp\big(A^{[k]}\big)$.

§B.3. Compound Equations: Equations Governing the Evolution of Volume Elements

Let $t \to A(t)$ be a $n \times n$ real matrix-valued function defined and continuous in **R**. We consider the linear system of ordinary differential equations

$$x'(t) = A(t)\,x(t). \tag{3.1}$$

Let $X(t)$ be its fundamental matrix such that $X(0) = I$, and $X(t)x_0, \cdots, X(t)x_k$, be k solutions of (3.1). We want to derive the differential equation their exterior product $y(t) = X(t)x_0 \wedge \cdots \wedge X(t)x_k$ satisfies.

Direct differentiation yields

$$\begin{aligned}\frac{d}{dt}y(t) &= \sum_{i=1}^{k} X(t)x_0 \wedge \cdots \wedge \frac{d}{dt}(X(t)x_i) \wedge \cdots \wedge X(t)x_k \\ &= \sum_{i=1}^{k} X(t)x_1 \wedge \cdots \wedge A(t)X(t)x_i \wedge \cdots \wedge X(t)x_k \\ &= A^{[k]}(t)\,X(t)x_1 \wedge \cdots \wedge X(t)x_k \;=\; A^{[k]}(t)\,y(t).\end{aligned}$$

This leads to the following important result.

Theorem B.3.1. *Suppose $x_1(t), \cdots, x_k(t)$ are solutions of the linear system (3.1). Then $y(t) = x_1(t) \wedge \cdots \wedge x_k(t)$ is a solution of the linear system*

$$y'(t) = A^{[k]}(t)\,y(t). \tag{3.2}$$

The system (3.2) is called the k-*th compound equation* of (3.1), with $A^{[k]}(t)$ being the k-th additive compound matrix of $A(t)$. Obviously, system (3.2) has $\binom{n}{k}$ equations.

Note that

$$X(t)x_0 \wedge \cdots \wedge X(t)x_k = X^{(k)}(t)\,x_1 \wedge \cdots \wedge x_k.$$

We have the following corollary of Theorem B.3.1.

Corollary B.3.2. *A fundamental matrix of the k-th compound equation* (3.2) *is given by* $X^{(k)}(t)$ *if* $X(t)$ *is a fundamental matrix of* (3.1).

Remarks.

(i) Suppose $\mathcal{K}$ is the oriented k-dimensional parallelepiped in $\mathbf{R}^n$ spanned by k linearly independent vectors $x_1, \cdots, x_k \in \mathbf{R}^n$. Let $\mathcal{K}$ evolve along the solutions of (3.1) and let $\mathcal{K}(t)$ be its position at time t. Then $\mathcal{K}(t)$ is spanned by $X(t)x_1, \cdots, X(t)x_k$ (see Figure B.3.1) and its volume is calculated as the norm of the vector

$$X(t)x_1 \wedge \cdots \wedge X(t)x_k,$$

which is a solution of the k-th compound equation (3.2). We thus reach the following conclusion:

> *the k-th compound equation* (3.2) *describes the evolution of any oriented k-dimensional parallelepiped under the linear system* (3.1).

(ii) When $k = 1$, the system (3.2) is identical to (3.1); when $k = n$, $A^{[n]}(t) = \text{tr}(A(t))$. Thus (3.2) becomes

$$y'(t) = \text{tr}(A(t))\, y(t), \tag{3.3}$$

for which a fundamental matrix is given by $X^{(n)}(t) = \det(X(t))$. Therefore the n-th compound equation (3.3) is the familiar equation of Liouville and Jacob (see [4]).

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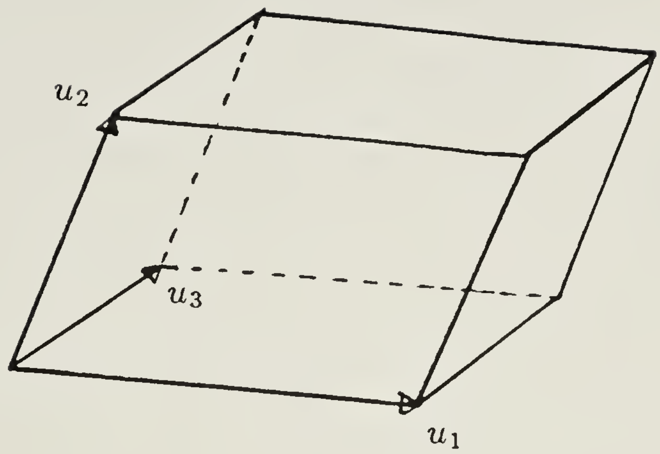


Figure B.1.1. A parallelopiped in $\mathbf{R}^3$.

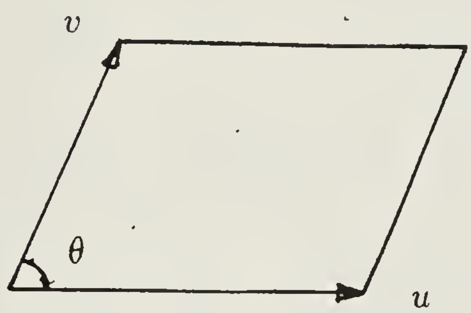
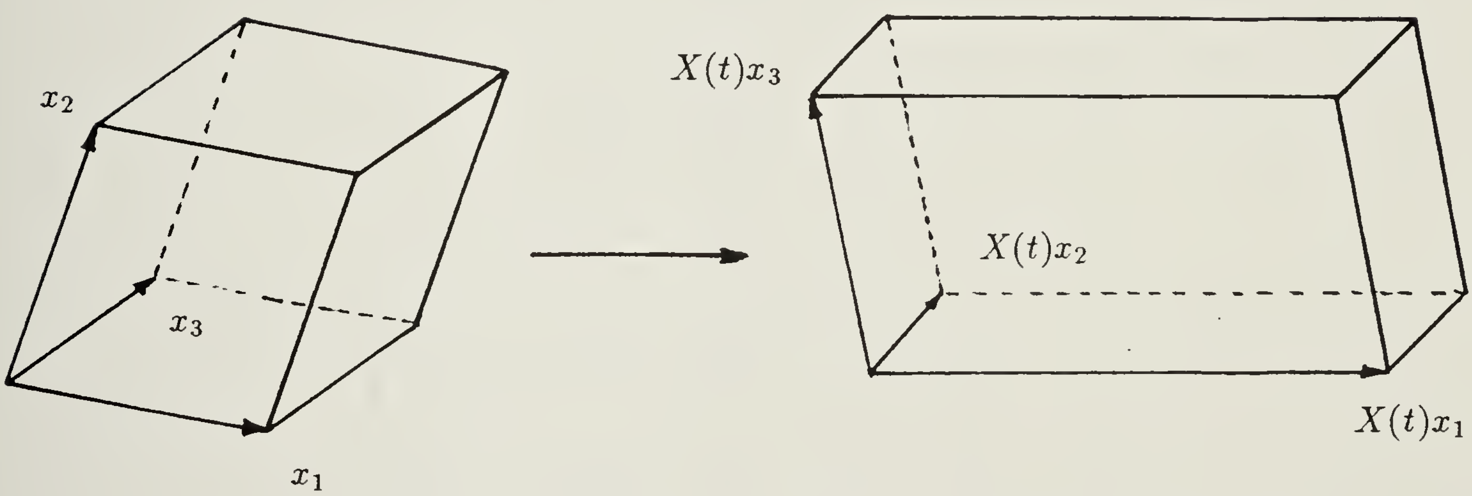


Figure B.1.2. A parallelogram in $\mathbf{R}^2$.



A parallelopiped $\mathcal{K}$.

The papallelopiped $\mathcal{K}(t)$.

Figure B.3.1. Evolution of a parallelopiped.

The matrices $A^{[k]}$ in the cases $n = 3, 4$ are as follows.

$n = 3$:

$$A^{[1]} = \begin{bmatrix} a_1^1 & a_1^2 & a_1^3 \\ a_2^1 & a_2^2 & a_2^3 \\ a_3^1 & a_3^2 & a_3^3 \end{bmatrix} = A$$

$$A^{[2]} = \begin{bmatrix} a_1^1+a_2^2 & a_2^3 & -a_1^3 \\ a_3^2 & a_1^1+a_3^3 & a_1^2 \\ -a_3^1 & a_2^1 & a_2^2+a_3^3 \end{bmatrix} \qquad \begin{matrix} (1) = (12) \\ (2) = (13) \\ (3) = (23) \end{matrix}$$

$$A^{[3]} = a_1^1 + a_2^2 + a_3^3 = \mathrm{Tr}\, A.$$

$n = 4$:

$$A^{[1]} = \begin{bmatrix} a_1^1 & a_1^2 & a_1^3 & a_1^4 \\ a_2^1 & a_2^2 & a_2^3 & a_2^4 \\ a_3^1 & a_3^2 & a_3^3 & a_3^4 \\ a_4^1 & a_4^2 & a_4^3 & a_4^4 \end{bmatrix} = A$$

Figure 3.2.2.

$$A^{[2]} = \begin{bmatrix} a^1_1+a^2_2 & a^3_2 & a^4_2 & -a^3_1 & -a^4_1 & 0 \\ a^2_3 & a^1_1+a^3_3 & a^4_3 & a^2_1 & 0 & -a^4_1 \\ a^2_4 & a^3_4 & a^1_1+a^4_4 & 0 & a^2_1 & a^3_1 \\ -a^1_3 & a^1_2 & 0 & a^2_2+a^3_3 & a^4_3 & -a^4_2 \\ -a^1_4 & 0 & a^1_2 & a^3_4 & a^2_2+a^4_4 & a^3_2 \\ 0 & -a^1_4 & a^1_3 & -a^2_4 & a^2_3 & a^3_3+a^4_4 \end{bmatrix} \begin{array}{l} (1) = (12) \\ (2) = (13) \\ (3) = (14) \\ (4) = (23) \\ (5) = (24) \\ (6) = (34) \end{array}$$

$$A^{[3]} = \begin{bmatrix} a^1_1+a^2_2+a^3_3 & a^4_3 & -a^4_2 & a^4_1 \\ a^3_4 & a^1_1+a^2_2+a^4_4 & a^3_2 & -a^3_1 \\ -a^2_4 & a^2_3 & a^1_1+a^3_3+a^4_4 & a^2_1 \\ a^1_4 & -a^1_3 & a^1_2 & a^2_2+a^3_3+a^4_4 \end{bmatrix} \begin{array}{l} (1) = (123) \\ (2) = (124) \\ (3) = (134) \\ (4) = (234) \end{array}$$

$$A^{[4]} = a^1_1 + a^2_2 + a^3_3 + a^4_4 = \mathrm{Tr}\, A.$$

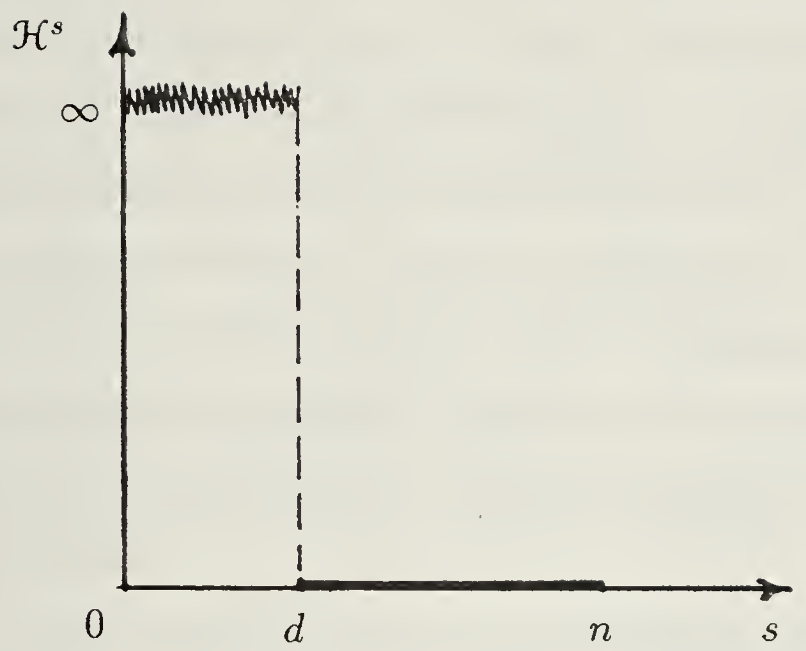


Figure C.2.1. Graph of $\mathcal{H}^s$ as a function of s.

APPENDIX C

HAUSDORFF MEASURE AND HAUSDORFF DIMENSION

Dimension is one of the main characteristics for complexity when we study the geometry of compact sets. Of many definitions of dimension in use, for example, Hausdorff dimension, fractal dimension, and Lyapunov dimension, etc., Hausdorff dimension is the oldest and the most important. It is mathematically convenient, as it is based on measures, which are relatively easy to manipulate. A major disadvantage is that in many cases it is hard to calculate or estimate either analytically or by computational methods. However, as many authors have pointed out, Hausdorff dimension is essential in the study of the geometry of irregular sets, such as fractal sets (see [3]).

In this appendix, we review the definitions and some simple properties of Hausdorff measures and Hausdorff dimension. More thorough treatments on these subjects may be found in [1] and [2].

§C.1 Hausdorff Measures

Let X be a normed linear space with norm $|\cdot|$. For a subset $U \subset X$, its *diameter* is defined as

$$|U| = \sup\{\, |x-y| \ : \ x, y \in U \,\}.$$

Obviously $|U|$ is finite if and only if U is bounded. Let $K \subset X$ be a compact set. A countable (or finite) open cover $\{U_i\}_i$ of K is said to be a *δ-cover* if $0 < |U_i| \leq \delta$ for each i.

Suppose s is a nonnegative number. For any $\delta > 0$ we define

$$\mathcal{H}^s_\delta(K) = \inf\{\, \sum_{i=1}^{\infty} |U_i|^s \ : \ \{U_i\}_i \text{ is a } \delta\text{-cover of } K \,\}. \tag{1.1}$$

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We can see from (1.1) that $\mathcal{H}^s_\delta(k)$ increases as δ decreases since the class of permissible covers of K is reduced. Therefore $\mathcal{H}^s_\delta(K)$ approaches a limit as $\delta \to 0$. We write

$$\mathcal{H}^s(K) = \lim_{\delta\to 0} \mathcal{H}^s_\delta(K) = \sup_{\delta>0} \mathcal{H}^s_\delta(K). \tag{1.2}$$

We call $\mathcal{H}^s(K)$ the s-dimensional *Hausdorff measure* of K. Obviously the value of $\mathcal{H}^s(K)$ could range from 0 to ∞.

It can be proved (see [2]) that $\mathcal{H}^s$ is a regular metric measure. Thus it has the following properties of measures.

Proposition C.1.1. *$\mathcal{H}^s$ enjoys the following properties:*

(1) $\mathcal{H}^s(\emptyset) = 0$.

(2) $\mathcal{H}^s(E) \leq \mathcal{H}^s(F)$ *if* $E \subset F$.

(3) $\mathcal{H}^s(\cup_{i=1}^\infty K_i) = \sum_{i=1}^\infty \mathcal{H}^s(K_i)$, *if* $\{K_i\}_{i=1}^\infty$ *is a family of disjoint Borel sets.*

Hausdorff measures generalize the usual ideas of length, area, and volume. It can be shown that, when $X = \mathbf{R}^n$, n-dimensional Hausdorff measure is a constant multiple of the n-dimensional Lebesgue measure. More precisely, if K is a Borel subset of $\mathbf{R}^n$ and $|\cdot|$ is the l_2 norm, then

$$\mathcal{H}^n(K) = c_n \mathcal{L}^n(K)$$

where $\mathcal{L}^n$ denotes the n-dimensional Lebesgue measure and the constant $c_n = \pi^{\frac{1}{2}n}/2^n(\frac{1}{2}n)!$ is the volume of the unit ball in $\mathbf{R}^n$. Similarly, for an integer $0 \leq m \leq n$, if K is a m-dimensional submanifold of $\mathbf{R}^n$, then $\mathcal{H}^m(K) = c_m \mathcal{L}^m(K)$. For example, $\mathcal{H}^0(E)$ is the number of points in E; $\mathcal{H}^1(C)$ is the length of a smooth curve C; $\mathcal{H}^2(S) = \frac{1}{4}\pi\,\mathrm{area}(S)$ if S is a smooth surface.

One of the important characteristics of the usual volume is its scaling property. That is, on magnification by a factor λ, the length of a curve will be multiplied by a factor λ, the area of a surface by a factor of λ^2, and the volume of a 3-dimensional body by a factor of λ^3. As the following proposition shows, Hausdorff measure also possesses the scaling property. A proof of this proposition can be found in [1].

Proposition C.1.2 (Scaling Property). *Suppose* $K \subset X$ *is a compact set and* $\lambda > 0$. *Then*

$$\mathcal{H}^s(\lambda K) = \lambda^s \mathcal{H}^s(K), \tag{1.3}$$

where $\lambda K = \{\lambda x : x \in K\}$.

A mapping $f : X \to Y$ from X to another normed linear space Y with norm $|\cdot|_1$ is said to be *Hölder continuous* if there exist constants $c > 0$ and $\alpha > 0$ such that

$$|f(x) - f(y)|_1 \leq c|x - y|^\alpha \tag{1.4}$$

for all $x, y \in X$. The constant α is called the Hölder exponent. When $\alpha = 1$, f is Lipschitz continuous. The following proposition describes how Hausdorff measures change under Hölder continuous mappings, a proof of which can be found in [1].

Proposition C.1.3. *Let* $K \subset X$ *be a compact set and the mapping* $f : X \to Y$ *satisfy* (1.4). *Then, for each* $s \geq 0$,

$$\mathcal{H}^{s/\alpha}\big(f(K)\big) \leq c^{s/\alpha} \mathcal{H}^s(K).$$

In particular, if f is Lipschitz continuous, namely f satisfies (1.4) with $\alpha = 1$, then

$$\mathcal{H}^s\big(f(K)\big) \leq c^s \mathcal{H}^s(K)$$

for each compact $K \subset X$. If f is an isometry, that is

$$|f(x) - f(y)|_1 = |x - y|,$$

then $\mathcal{H}^s(f(K)) = \mathcal{H}^s(K)$. Therefore Hausdorff measure is preserved under isometrics. Since a translation $x \mapsto x + a$ is an isometry, we have the following corollary.

Corollary C.1.4. *$\mathcal{H}^s$ is translation invariant; namely*

$$\mathcal{H}^s(K + a) = \mathcal{H}^s(K)$$

for all compact K and $a \in X$, where $K + a = \{x + a \,;\, x \in K\}$.

In the case when X is an inner product space with inner product $\langle , \rangle$ and the norm $|\cdot|$ is given by $|x| = \langle x, x\rangle^{1/2}$, since the rotation in X is an isometry, $\mathcal{H}^s$ is also rotation invariant.

§C.2. Hausdorff Dimension

Hausdorff measure $\mathcal{H}^s$ is defined for all $s \geq 0$. It turns out, however, that $\mathcal{H}^s(K)$ is either zero or infinity for most s. In fact, for two real numbers $0 \leq s < t$ and a δ-cover $\{U_i\}_i$ of K, we have from (1.1) that

$$\sum_i |U_i|^t \leq \delta^{t-s} \sum_i |U_i|^s$$

which leads to $\mathcal{H}^t_\delta(K) \leq \delta^{t-s}\mathcal{H}^s_\delta(K)$ for all $\delta > 0$. Letting $\delta \to 0$, we see that if $\mathcal{H}^s(K) < \infty$ then $\mathcal{H}^t(K) = 0$ for all $t > s$. A graph of $\mathcal{H}^s(K)$ against s is shown in the Figure C.2.1 . Observe that there is a $0 \leq d \leq \infty$ at which $\mathcal{H}^s(K)$ jumps from ∞ to 0. This suggests that d be considered as a dimension of K.

The *Hausdorff dimension* of a compact set K is defined as

$$\dim_H K = \inf\{\, s \;:\; \mathcal{H}^s(K) = 0\,\} = \sup\{\, s \;:\; \mathcal{H}^s(K) = +\infty\,\}. \tag{2.1}$$

We can see from the definition that it is possible that $\dim_H K = \infty$. In this case $\mathcal{H}^s(K) = \infty$ for all $s \geq 0$. Suppose $\dim_H K < \infty$. Then

$$\mathcal{H}^s(K) = \begin{cases} \infty, & \text{if } s < \dim_H K \\ 0, & \text{if } s > \dim_H K. \end{cases}$$

However, when $d = \dim_H K$, $\mathcal{H}^d(K)$ may be $0, +\infty$, or may satisfy $0 < \mathcal{H}^d(K) < \infty$.

As a simple example, let C be the unit circle in $\mathbf{R}^2$. We know that $\mathcal{H}^1(C) = \mathcal{L}^1(C) = 2\pi$. Thus $\dim_H K = 1$ and $\mathcal{H}^s(C) = \infty$ if $s < 1$ and $\mathcal{H}^s(C) = 0$ if $s > 1$, which is what we would expect. Similarly, a k-dimensional submanifold of $\mathbf{R}^n$ has Hausdorff dimension k.

Some fractal sets have fractional Hausdorff dimensions. For example, the Cantor middle third set in $[0, 1]$ has Hausdorff dimension $d = \log 2/\log 3$.

The following are some properties of Hausdorff dimension.

(i) *monotonicity.* If $E \subset F$ then $\dim_H E \leq \dim_H F$. This follows from the property of Hausdorff measure that $\mathcal{H}^s(E) \leq \mathcal{H}^s(F)$ for each $s \geq 0$.

(ii) *countable stability.* If $\{K^n\}_{n=1}^{\infty}$ is a (countable) family of compact sets, then

$$\dim_H \cup_{n=1}^{\infty} K_n = \sup_n \{ \dim_H K_n \}.$$

It follows from the monotonicity property (ii) that $\dim_H \cup_{n=1}^{\infty} K_n \geq \dim_H K_n$ for each n. On the other hand, the measure property $\mathcal{H}^s(\cup_{n=1}^{\infty}) \leq \sum_{n=1}^{\infty} \mathcal{H}^s(K_n)$ for all $s \geq 0$ gives the opposite inequality.

These simple properties can be used to find the Hausdorff dimension of some sets. For example, if K is a countable set, then from the countable stability $\dim_H K = 0$ since the Hausdorff dimension of a single point is obviously 0. In particular, the Hausdorff dimension of the set $\mathbf{Q}$ of rationals in $\mathbf{R}$ is 0, even though $\mathbf{Q}$ is dense in $\mathbf{R}$.

The following proposition which shows how Hausdorff dimension changes under Hölder continuous mappings follows from the corresponding property for Hausdorff measure given in Proposition C.1.3.

Proposition C.2.2. *Suppose* $K \subset X$ *is compact and the mapping* $f : X \to Y$ *satisfies a Hölder condition on* K

$$|f(x) - f(y)|_1 \leq c\,|x - y|^{\alpha} \qquad x, y \in K.$$

Then

$$\dim_H f(K) \leq \frac{1}{\alpha} \dim_H K.$$

As a special case we have the following corollary.

Corollary C.2.3.

(1) *If* $f : X \to Y$ *is a Lipschitz continuous mapping, then* $\dim_H f(K) \leq \dim_H K$ *for each compact* $K \subset X$.

(2) *If* $f : X \to Y$ *is a bi-Lipschitz mapping, namely*

$$c_1 |x - y| \leq |f(x) - f(y)|_1 \leq c_2 |x - y| \qquad x, y \in X$$

where $0 < c_1 \leq c_2 < \infty$, *then* $\dim_H f(K) = \dim_H K$.

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APPENDIX D

AN ELEMENTARY PROOF OF THE CLOSING LEMMA

The purpose of this appendix is to give an elementary proof of a restricted version of the C^1 Closing Lemma which is needed in the thesis.

In the section §D.1, a local C^1 Closing Lemma for general autonomous system is formulated and proved. In §D.2, a similar result is proved for autonomous systems having an invariant linear subspace discussed in Chapter VI.

§D.1. A Local Closing Lemma for General Autonomous Systems

We consider an autonomous system in $\mathbf{R}^n$

$$x' = f(x) \tag{1.1}$$

where the function $x \mapsto f(x) \in \mathbf{R}^n$ is C^1 for x in an open subset D of $\mathbf{R}^n$. We formulate in the following a local version of the C^1-Closing Lemma of Pugh which plays an essential role in the development of Chapter V.

Let $|\cdot|$ denote a vector norm on $\mathbf{R}^n$ and the operator norm it induces for linear mappings from $\mathbf{R}^n$ to $\mathbf{R}^n$. The distance between two functions $f, g \in C^1(D \to \mathbf{R}^n)$ such that $f - g$ has compact support is

$$|f-g| = \sup\{|f(x) - g(x)| + |Df(x) - Dg(x)| \,:\, x \in D\}. \tag{1.2}$$

A function $g \in C^1(D \to \mathbf{R}^n)$ is called a C^1 *local ϵ-perturbation* of f at $x_0 \in D$ if there exists an open neighbourhood U of x_0 in D such that $\operatorname{supp}(f-g) \subset U$ and $|f-g| < \epsilon$. For such a g we consider the corresponding differential equation

$$x' = g(x). \tag{1.3}$$

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Lemma D.1.1. (Local $\mathbf{C}^1$ Closing Lemma). *Let $f \in C^1(D \to \mathbf{R}^n)$. Suppose that x_0 is a nonwandering point for (1.1) and that $f(x_0) \neq 0$. Then, for each neighbourhood U of x_0 and $\epsilon > 0$, there exists a C^1 local ϵ-perturbation g of f at x_0 such that*

(1) $\operatorname{supp}(f-g) \subset U$, *and*

(2) *the system* (1.3) *has a nonconstant periodic solution whose trajectory intersects U.*

Proof. We first observe that, if $x \mapsto F(x)$ is a differentiable function with bounded convex domain $V \subset \mathbf{R}^m$ and range in $\mathbf{R}^m$, then the Mean Value Theorem gives

$$F_i(b) - F_i(a) = \sum_{j=1}^{m} \frac{\partial F_i}{\partial x_j}(c^i)(b_j - a_j), \quad i = 1, \ldots, m$$

for some points $c^1, \ldots, c^m$ on the line segment joining a and b. It follows that, if the matrix $[\frac{\partial F_i}{\partial x_j}(c^i)]$ is nonsingular for all sets of m colinear points $c^i \in V$, $i = 1, \ldots, m$, F is one-to-one. In particular, F is one-to-one if it is sufficiently C^1-close to the identity map $x \mapsto x$ on V.

If $p \in \mathbf{R}^{n-1}$ and $\rho > 0$, let $|p| = (p^*p)^{1/2}$ and $B_\rho = \{p : |p| < \rho\}$. For $p,\ p_i \in \overline{B}_{2\rho}$, let the C^1 function $p \mapsto \tau_i(p)$ be defined by

$$\begin{aligned} \tau_i(p) &= p, & \rho \leq |p| \leq 2\rho \\ \tau_i(p) &= p + p_i(\rho^2 - |p|^2)^2/\rho^4, & |p| \leq \rho. \end{aligned} \tag{1.4}$$

Evidently $\tau_i(0) = p_i$ and τ_i is a diffeomorphism if $|p_i|$ is sufficiently small since this implies τ_i is C^1-close to the identity. Also $\tau_i(\overline{B}_{2\rho}) = \overline{B}_{2\rho}$ if $|p_i|$ is small. This implies that

$$\tau_{12} = \tau_1 \circ \tau_2^{-1} : \overline{B}_{2\rho} \to \overline{B}_{2\rho} \tag{1.5}$$

is a diffeomorphism which satisfies $\tau_{12}(p_2) = p_1$, and which is close to the identity map if p_1, p_2 are sufficiently close to 0.

Next, if $\delta > 0$ and $t \in [-2\delta, 2\delta]$, define $t \mapsto \lambda(t)$ by

$$\begin{aligned} &\lambda(t) = 0, && -2\delta \le t < -\delta \\ &\lambda(t) = (t+\delta)^3(1/\delta^3 - 3t/\delta^4 + 6t^2/\delta^5), && -\delta \le t \le 0 \\ &\lambda(t) = 1, && 0 < t \le 2\delta. \end{aligned} \tag{1.6}$$

It is easily checked that λ is of class C^2 on $[-2\delta, 2\delta]$, $\lambda'(t) = 30(t+\delta)^2 t^2/\delta^5$, $-\delta \le t \le 0$ and therefore

$$0 \le \lambda(t) \le 1, \quad 0 \le \lambda'(t) \le 15/8\delta, \quad -2\delta < t < 2\delta. \tag{1.7}$$

Without loss of generality, we may assume that 0 is the nonwandering point in the Closing Lemma. Since 0 is not an equilibrium, we may also assume that the disc $\{0\} \times \overline{B}_{2\rho}$ in the hyperplane $x_1 = 0$ is transversal to the flow of (1.1). If $|p| \le 2\rho$, let $\varphi(t,p) = x\big(t,(0,p)\big)$. For δ sufficiently small, φ is a diffeomorphism on $\overline{B}_{2\delta,2\rho}$, where $B_{\delta,\rho} = (-\delta,\delta) \times B_\rho$. We may also assume that $\varphi(\overline{B}_{2\delta,2\rho}) \subset U$.

Now consider ψ defined by

$$\psi(t,p) = \varphi(t,p) + \lambda(t)\big[\varphi\big(t,\tau_{12}(p)\big) - \varphi(t,p)\big] \tag{1.8}$$

if $(t,p) \in \overline{B}_{2\delta,2\rho}$. We wish to show that ψ is a diffeomorphism with $\psi(\overline{B}_{2\delta,2\rho}) = \varphi(\overline{B}_{2\delta,2\rho})$ when $|p_1|, |p_2|$ are small. From (1.4), (1.5), (1.6), (1.7), we find $\psi(\overline{B}_{2\delta,2\rho} \backslash \overline{B}_{\delta,\rho}) = \varphi(\overline{B}_{2\delta,2\rho} \backslash \overline{B}_{\delta,\rho})$ and, since ψ is C^1-close to φ if p_1, p_2 are close to 0, $\psi(B_{2\delta,2\rho}) \subset \varphi(B_{2\delta,2\rho})$ and $\psi(\partial B_{2\delta,2\rho}) = \varphi(\partial B_{2\delta,2\rho}) = \partial\varphi(B_{2\delta,2\rho})$. When $|p_1|, |p_2|$ are small, $\varphi^{-1} \circ \psi$ is C^1-close to the identity and therefore a diffeomorphism; the preceding discussion implies $\varphi^{-1} \circ \psi(B_{2\delta,2\rho}) \subset B_{2\delta,2\rho}$, $\varphi^{-1} \circ \psi(\partial B_{2\delta,2\rho}) = \partial B_{2\delta,2\rho}$. The Invariance of Domain Theorem ([1], p. 50) implies the complement of $\varphi^{-1} \circ \psi(B_{2\delta,2\rho})$ does not intersect $B_{2\delta,2\rho}$ and thus $\varphi^{-1} \circ \psi(\overline{B}_{2\delta,2\rho}) = \overline{B}_{2\delta,2\rho}$. Therefore $\psi(\overline{B}_{2\delta,2\rho}) = \varphi(\overline{B}_{2\delta,2\rho})$ and ψ is a diffeomorphism when p_1 and p_2 are close to 0 as asserted.

Define $x \mapsto g(x)$ by

$$\begin{aligned} g(x) &= f(x), && x \in D \backslash \psi(B_{2\delta,2\rho}) \\ g(x) &= \frac{\partial}{\partial t}\,\psi(t,p), && x = \psi(t,p), \quad (t,p) \in B_{2\delta,2\rho}. \end{aligned} \tag{1.9}$$

For each $x_0 = \psi(t_0,p_0) \in \psi(B_{2\delta,2\rho})$ the equation (1.3) has a solution $x(t) = \psi(t,p_0)$,
$|t| \leq 2\delta$, which is also a solution of (1.1) for $-2\delta \leq t \leq -\delta$ and $0 \leq t \leq 2\delta$ and may be continued as a solution of both (1.1) and (1.3) for $|t| \geq \delta$. Since $g(x) = f(x)$ if $x \in D\backslash\psi(B_{\delta,\rho})$, to complete the proof of Lemma 2.7 it suffices to prove that g is of class C^1 on $\psi(B_{2\delta,2\rho})$, is C^1-close to f on $\psi(B_{2\delta,2\rho})$ if $|p_1|$, $|p_2|$ are small and that p_1, p_2 may be chosen so that (1.3) has a periodic trajectory which intersects $\psi(B_{2\delta,2\rho}) = \varphi(B_{2\delta,2\rho})$.

From (1.9), if $x \in \psi(B_{2\delta,2\rho})$,

$$g(x) = G \circ \psi^{-1}(x), \qquad Dg(x) = DG \circ D\psi^{-1}(x),$$

where

$$\begin{aligned} &G(t,p) = \\ &f(\varphi(t,p)) + \lambda(t)\big[f(\varphi(t,\tau_{12}(p)) - f(\varphi(t,p))\big] + \lambda'(t)\big[\varphi(t,\tau_{12}(p)) - \varphi(t,p)\big]. \end{aligned} \tag{1.10}$$

Similarly

$$f(x) = F \circ \varphi^{-1}(x), \qquad Df(x) = DF \circ D\varphi^{-1}(x),$$

where

$$F(t,p) = f(\varphi(t,p)).$$

Thus $g \in C^1$ and G, ψ^{-1} and their partial derivatives are uniformly close to the corresponding expressions for F, φ^{-1} respectively if p_1, p_2 are close to 0. Thus g has the desired approximation property with respect to f.

To prove the assertion about a periodic trajectory, since 0 is nonwandering and $\{0\} \times B_{2\rho}$ is transversal to the flow of (1.1) at 0, there exist $p_1, p_2 \in B_{2\rho}$

arbitrarily close to 0 such that $(0, p_1) = \varphi(-T, p_2)$ for some $T > 0$. Now

$$x(t) = \varphi(t, p_2), \quad -T \le t \le -2\delta$$

$$x(t) = \psi(t, p_2), \quad -2\delta \le t \le 0$$

where the points p_1, p_2 determine τ_{12} in (1.5), (1.8), is a solution of (1.3) which satisfies

$$x(0) = \varphi(0, p_2) + \lambda(0)[\varphi(0, p_1) - \varphi(0, p_2)] = \varphi(0, p_1) = (0, p_1) = x(-T)$$

and is therefore periodic of period T; see Figure D.1.1.

The argument given here may also be modified to obtain an equation (1.3) with a periodic solution whose trajectory contains the nonwandering point 0.

§D.2. Autonomous Systems with an Invariant Affine Manifold

In this section, we show that Lemma D.1.1 also holds for autonomous systems satisfying $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$ in §6.2 of Chapter VI. More precisely, let B and Γ be the constant matrix and the invariant affine manifold determined by $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$, we prove the following result.

Lemma D.2.1. *Let $f \in C^1(D \to \mathbf{R}^n)$ be a vector field satisfying $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$. Suppose that $x_0 \in \Gamma$ is a nonwandering point for (2.1) and that $f(x_0) \neq 0$. Then for each neighbourhood U of x_0 and $\epsilon > 0$, there exists a C^1 ϵ-perturbation of f, $g \in C^1(D \to \mathbf{R}^n)$, such that*

(1) *supp$(f - g) \subset U$,*

(2) *g also satisfies $(\mathbf{H}_1')$ and $(\mathbf{H}_2')$ with the same matrix B,*

(3) *the system (1.3) has a nonconstant periodic solution whose trajectory intersects U.*

Remark. As we have noted in the Lemma 6.2.8 of §6.2, we only require that g satisfy the weaker assumptions $(\mathbf{H}_1')$ and $(\mathbf{H}_2')$. This version of the local closing lemma is sufficient for deriving Theorem 6.2.9 of §6.2, Chapter VI.

Proof. Let g be the ϵ-perturbation of f obtained by Lemma D.1.1. Then g satisfies (1) and (3). It remains to show that g satisfies (2).

We claim the following:

(a). the mapping $\tau_{12} : \bar{B}_{2\rho} \to \bar{B}_{2\rho}$ defined in (1.5) preserves the affine manifold Γ.

(b). $Bg(x) = 0$ on Γ.

As in the proof of Lemma D.1.1, we may assume that $0 \in \Gamma$ is the nonwandering point and thus Γ is linear. Since τ_i preserves lines through p_i, $i = 1, 2$, we can see that (a) follows from the definition of τ_{12} in (1.5). To see (b), note that $g(x) = G \circ \psi^{-1}(x)$ for $x = \psi(t,p) \in B_{2\delta,2\rho}$ and G is given in (1.10). Using the fact that

$$Bf(x) = 0, \qquad \text{and} \qquad B\varphi(t,p) = \text{constant} \tag{2.1}$$

for $(t,p) = \psi^{-1}(x)$ and $x \in \Gamma$, we can see from (1.10) that $BG(t,p) = 0$ for $(t,p) \in B_{2\delta,2\rho}$, and $p \in \Gamma$.

Now for $(t,p) \in B_{2\delta,2\rho}$, and $p \in \Gamma$, let D_t and D_p denote the partial derivative with respect to t and p, respectively, and $D_{t,p} = (D_t, D_p)$. We have

$$\begin{aligned} BD_tG(t,p) &= BDxfD_t\varphi(t,p) + \lambda(t)[BD_xfD_t\varphi(t,\tau_{12}(p)) \\ &\quad - BD_xfD_t\varphi(t,p)] \\ &= -\nu B\left[D_t\varphi(t,p) + \lambda(t)(D_t\varphi(t,\tau_{12}(p)) - D_t\varphi(t,p))\right] \\ &= -\nu BD_t\psi(t,p), \end{aligned}$$

and

$$\begin{aligned} BD_pG(t,p) &= BDxfD_p\varphi(t,p) + \lambda(t)[BD_xfD_p\varphi(t,\tau_{12}(p))D\tau_{12}(p) \\ &\quad - BD_xfD_p\varphi(t,p)] \\ &= -\nu B\left[D_p\varphi(t,p) + \lambda(t)(D_p\varphi(t,\tau_{12}(p))D\tau_{12}(p) - D_p\varphi(t,p))\right] \\ &= -\nu BD_p\psi(t,p). \end{aligned}$$

Therefore $BD_{t,p}G(t,p) = -\nu BD_{t,p}\psi(t,p)$. This implies

$$BDg(x) = BDG \circ D\psi^{-1}(x) = -\nu B,$$

for all $x = \psi(t,p)$, $(t,p) \in B_{2\delta,2\rho}$, and $p \in \Gamma$. Obviously this holds for $x \in \Gamma \cap D \setminus \psi(B_{2\delta,2\rho})$ trivially. Therefore the lemma is proved. □

§D.2. Bibliography for Appendix D

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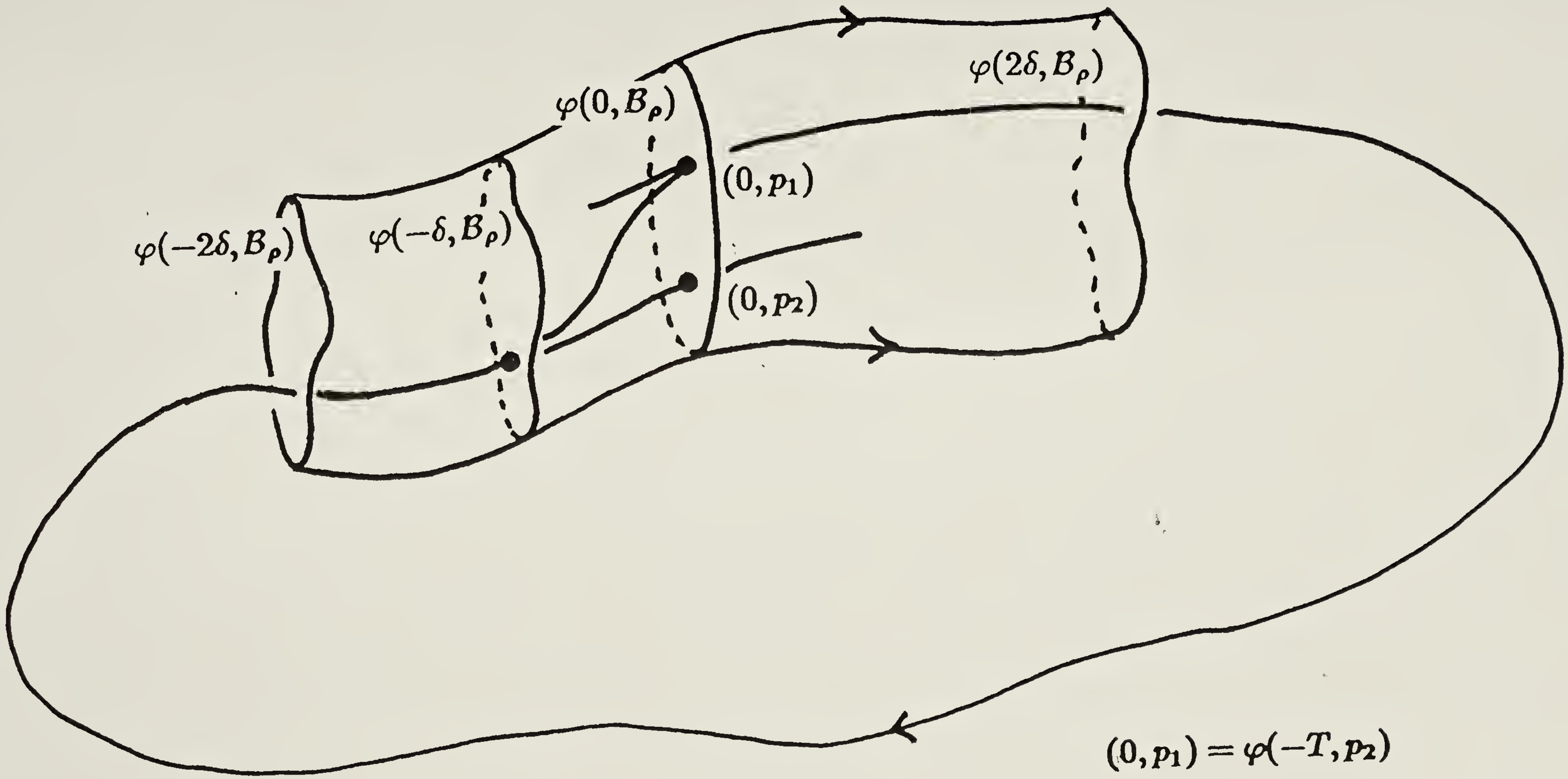

Figure D.1.1. Closing up an orbit.